H2 MATHEMATICS **9758/01**

Paper 1

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3 hours

Proposed Solutions

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1 From the first equation,

$$w = \frac{-1 - \mathrm{i}z}{2}.$$

Substituting into the second equation,

$$\begin{aligned}
(2-\mathrm{i})z + \mathrm{i}\cdot\frac{-1-\mathrm{i}z}{2} &= 6 \quad \text{M1}\\
2(2-\mathrm{i})z + (-\mathrm{i}+z) &= 12\\
(4-2\mathrm{i}+1)z &= 12+\mathrm{i}\\
z &= \frac{12+\mathrm{i}}{5-2\mathrm{i}}\\
&= \frac{12+\mathrm{i}}{5-2\mathrm{i}}\cdot\frac{5+2\mathrm{i}}{5+2\mathrm{i}} \quad \text{M1}\\
&= \frac{(60-2)+(24+29)\mathrm{i}}{5^2+2^2}\\
&= \frac{58+29\mathrm{i}}{29} = 2+\mathrm{i}. \quad \text{A1}
\end{aligned}$$

Back-substituting,

$$w = \frac{1}{2}(-1-\mathrm{i}(2+\mathrm{i})) = \frac{1}{2}(-2\mathrm{i}) = -\mathrm{i}. \quad \text{A1}$$

2 **(a)** Differentiating twice,

$$\begin{aligned}
\mathrm{f}'(x) &= \frac{1}{1+(\sqrt{2}+x)^2}, \quad \text{M1}\\
\mathrm{f}''(x) &= -\frac{2(\sqrt{2}+x)}{\left(1+(\sqrt{2}+x)^2\right)^2}. \quad \text{M1} \quad \text{A1}
\end{aligned}$$

(b) Setting $x = 0$,

$$\begin{aligned}
\mathrm{f}(0) &= \tan^{-1}(\sqrt{2}),\\
\mathrm{f}'(0) &= \frac{1}{1+(\sqrt{2})^2} = \frac{1}{3},\\
\mathrm{f}''(0) &= \frac{2\sqrt{2}}{\left(1+(\sqrt{2})^2\right)^2} = \frac{2}{9}\sqrt{2}. \quad \text{M1}
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathrm{f}(x) &\approx \tan^{-1}(\sqrt{2}) + \frac{1}{3}x + \frac{1}{9}\sqrt{2}x^2 \quad \text{M1}\\
&= 0.95531 + 0.33333x + 0.15713x^2\\
&\approx 0.955 + 0.333x + 0.157x^2. \quad \text{(3sf)} \quad \text{A1}
\end{aligned}$$

3 **(a)** By the chain rule,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{\mathrm{d}y}{\mathrm{d}x}\cdot\frac{\mathrm{d}x}{\mathrm{d}t}\\ \frac{1}{2}(3\mathrm{e}^{3t}+6\mathrm{e}^{-3t}) &= \frac{\mathrm{d}y}{\mathrm{d}x}\cdot\left(\frac{1}{2}(3\mathrm{e}^{3t}-6\mathrm{e}^{-3t})\right). \quad \text{M1}\end{aligned}$$

When $t=\frac{1}{3}\ln 2$, $\mathrm{e}^{-3t}=\mathrm{e}^{-\ln 2}=\frac{1}{2}$. Similarly, $\mathrm{e}^{3t}=2$. Hence,

$$\begin{aligned}\frac{1}{2}\left(3\cdot 2+6\cdot\frac{1}{2}\right) &= \frac{\mathrm{d}y}{\mathrm{d}x}\cdot\left(\frac{1}{2}\left(3\cdot 2-6\cdot\frac{1}{2}\right)\right), \quad \text{M1}\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= 3.\end{aligned}$$

Thus, the gradient of the normal is $-\frac{1}{3}$. A1

(b) We observe that

$$\begin{aligned}x+y &= \mathrm{e}^{3t},\\ x-y &= 2\mathrm{e}^{-3t},\\ (x+y)(x-y) &= \mathrm{e}^{3t}\cdot 2\mathrm{e}^{-3t} \quad \text{M1}\\ x^2-y^2 &= 2. \quad \text{A1}\end{aligned}$$

Restriction on x: $x\geqslant\sqrt{2}$. A1

4 **(a)** By the chain rule,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\cot x) &= \frac{\mathrm{d}}{\mathrm{d}x}\tan\left(\frac{1}{2}\pi-x\right)\\ &= \sec^2\left(\frac{1}{2}\pi-x\right)\cdot -1 \quad \text{M1}\\ &= -\mathrm{cosec}^2 x. \quad \text{M1}\end{aligned}$$

(b) By double angle formulae,

$$\begin{aligned}\sin 2x\tan x &= 2\sin x\cos x\cdot\frac{\sin x}{\cos x}\\ &= 2\sin^2 x. \quad \text{B1}\end{aligned}$$

(c) We first remark that

$$\begin{aligned}\mathrm{cosec}\,6x\cot 3x &= \frac{1}{\sin 6x\tan 3x}\\ &= \frac{1}{2\sin^2 3x}\\ &= \frac{1}{2}\mathrm{cosec}^2 3x. \quad \text{M1}\end{aligned}$$

Hence,

$$\begin{aligned}
\int_{\frac{1}{18}\pi}^{\frac{1}{9}\pi} \operatorname{cosec} 6x \cot 3x \,\mathrm{d}x &= -\frac{1}{6}\int_{\frac{1}{18}\pi}^{\frac{1}{9}\pi} 3 \cdot -\operatorname{cosec}^2 3x \,\mathrm{d}x \quad \text{M1}\\
&= -\frac{1}{6}\left[\cot 3x\right]_{\frac{1}{18}\pi}^{\frac{1}{9}\pi}\\
&= -\frac{1}{6}\left(\cot\frac{1}{3}\pi - \cot\frac{1}{6}\pi\right) \quad \text{M1}\\
&= -\frac{1}{6}\left(\frac{1}{\sqrt{3}} - \sqrt{3}\right)\\
&= -\frac{1}{6}\cdot -\frac{2}{3}\sqrt{3}\\
&= \frac{1}{9}\sqrt{3}. \quad \text{A1}
\end{aligned}$$

5 **(a)** When the line intersects the curve,

$$\begin{aligned}
(x+8)^2 + (mx-14)^2 &= 52\\
x^2 + 16x + 64 + m^2x^2 - 28mx + 196 &= 52 \quad \text{M1}\\
(1+m^2)x^2 + (16-28m)x + 208 &= 0.
\end{aligned}$$

Since the line is a tangent, the discriminant equals 0:

$$\begin{aligned}
(16-28m)^2 - 4(1+m^2)(208) &= 0 \quad \text{M1}\\
16(4-7m)^2 - 16(52+52m^2) &= 0\\
(4-7m)^2 - (52+52m^2) &= 0\\
16 - 56m + 49m^2 - (52+52m^2) &= 0 \quad \text{M1}\\
-3m^2 - 56m - 36 &= 0\\
3m^2 + 56m + 36 &= 0. \quad \text{M1}
\end{aligned}$$

(b) The tangents are given by $y = mx$, where m satisfies **(a)**. Using G.C., $m = -\frac{2}{3}, -18$. M1
For each value of m, since (discriminant) $= 0$,

$$x = \frac{-(16-28m) \pm \sqrt{(\text{discriminant})}}{2(1+m^2)} = \frac{14m-8}{1+m^2}. \quad \text{M1}$$

Setting $m = -\frac{2}{3}$, $x = -12$ and $y = -\frac{2}{3} \cdot -12 = 8$. M1
Setting $m = -18$, $x = -\frac{4}{5}$ and $y = -18 \cdot -\frac{4}{5} = \frac{72}{5}$.
Thus, without loss of generality, $A(-12, 8)$ and $B(-\frac{4}{5}, \frac{72}{5})$. A1

6 **(a)** By long division,

$$\mathrm{f}(x) = \frac{ax+k}{x-a} = a + \frac{a^2+k}{x-a}. \quad \text{M1}$$

Thus, we make the transformations

$$y = \frac{1}{x} \quad \longrightarrow \quad y = \frac{1}{x-a} \quad \longrightarrow \quad y = \frac{a^2+k}{x-a} \quad \longrightarrow \quad y = a + \frac{a^2+k}{x-a}.$$

In description:

(A) Translate a units in the positive x-direction. A1
(B) Scale by factor $a^2 + k$ parallel to the y-axis. A1
(C) Translate a units in the positive y-direction. A1

(b) Write $y = \mathrm{f}^{-1}(x)$. Then

$$\begin{aligned} x &= \mathrm{f}(y) \\ &= \frac{ay+k}{y-a} \\ xy - ax &= ay + k \quad \text{M1} \\ xy - ay &= ax + k \\ y &= \frac{ax+k}{x-a} \\ \mathrm{f}^{-1}(x) &= \frac{ax+k}{x-a}. \quad \text{A1} \end{aligned}$$

Remark: A function $y = \dfrac{ax+b}{cx+d}$ is **always** *self-inverse*, that is, $\mathrm{f}^{-1}(x) = \mathrm{f}(x)$, when $a = -d$.

(c) By **(b)**, $\mathrm{f}^{-1}(x) = \mathrm{f}(x)$.
Hence, $\mathrm{f}^2(x) = \mathrm{f}(\mathrm{f}^{-1}(x)) = x$. A1

(d) We have

$$\begin{aligned} \mathrm{f}^{2023}(1) &= \mathrm{f}^{2\cdot1011+1}(1) \\ &= \mathrm{f}((\mathrm{f}^2)^{1011}(1)) \quad \text{M1} \\ &= \mathrm{f}(1) \\ &= \frac{a+k}{1-a}. \quad \text{A1} \end{aligned}$$

7 **(a)** By the product rule,

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= -3x^{-4}\ln x + x^{-3}\cdot\frac{1}{x} \quad \text{M1} \\ &= -\frac{3\ln x}{x^4} + \frac{1}{x^4} \\ &= \frac{1-3\ln x}{x^4}. \end{aligned}$$

At the turning point, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$, thus

$$\begin{aligned} \frac{1-3\ln x}{x^4} &= 0 \quad \text{M1} \\ 1 - 3\ln x &= 0 \\ \ln x &= \frac{1}{3} \\ x &= \mathrm{e}^{\frac{1}{3}} \quad \text{M1} \\ y &= \left(\mathrm{e}^{\frac{1}{3}}\right)^{-3}\cdot\frac{1}{3} = \frac{1}{3\mathrm{e}}. \end{aligned}$$

Thus the turning point has coordinates $\left(\mathrm{e}^{\frac{1}{3}}, \dfrac{1}{3\mathrm{e}}\right)$. A1

(b) We first find

$$\begin{aligned} \int x^{-3}\ln x\,\mathrm{d}x &= \underbrace{-\frac{1}{2}x^{-2}}_{\text{I}}\,\underbrace{\ln x}_{\text{S}} - \int \underbrace{-\frac{1}{2}x^{-2}}_{\text{I}}\,\underbrace{\frac{1}{x}}_{\text{D}}\,\mathrm{d}x \quad \text{M1} \\ &= -\frac{1}{2}x^{-2}\ln x + \frac{1}{2}\int x^{-3}\,\mathrm{d}x \\ &= -\frac{1}{2}x^{-2}\ln x - \frac{1}{4}x^{-2} + C. \quad \text{M1} \end{aligned}$$

Thus, the required area is

$$\begin{aligned}\int_1^3 |y|\,\mathrm{d}x &= \int_1^3 x^{-3}\ln x\,\mathrm{d}x \\ &= \left[-\frac{1}{2}x^{-2}\ln x - \frac{1}{4}x^{-2}\right]_1^3 \quad \text{M1} \\ &= \left(-\frac{1}{18}\ln 3 - \frac{1}{36}\right) - \left(-\frac{1}{4}\right) \quad \text{M1} \\ &= \left(\frac{2}{9} - \frac{1}{18}\ln 3\right) \text{units}^2. \quad \text{A1}\end{aligned}$$

8 **(a)** Write $2x - 1 = A(2x+2) + B$ for constants A and B to be determined. Then

$$\begin{aligned}\int \frac{2x-1}{x^2+2x+1}\,\mathrm{d}x &= A\int \frac{2x+2}{x^2+2x+1}\,\mathrm{d}x + B\int \frac{1}{(x+1)^2}\,\mathrm{d}x \quad \text{M1} \\ &= A\ln|x^2+2x+1| - B\cdot\frac{1}{x+1} + C. \quad \text{M1}\end{aligned}$$

Comparing coefficients, $A = 1$ and $B = -3$. M1
Hence,

$$\int \frac{2x-1}{x^2+2x+1}\,\mathrm{d}x = \ln|x^2+2x+1| + \frac{3}{x+1} + C. \quad \text{A1}$$

(b) Denote $\mathrm{F}(x) = \ln|x^2+2x+1| + \dfrac{3}{x+1}$. Then

$$\begin{aligned}\int_0^2 \frac{|2x-1|}{x^2+2x+1}\,\mathrm{d}x &= -\int_0^{\frac{1}{2}} \frac{2x-1}{x^2+2x+1}\,\mathrm{d}x + \int_{\frac{1}{2}}^2 \frac{2x-1}{x^2+2x+1}\,\mathrm{d}x \quad \text{M1} \\ &= -(\mathrm{F}(\tfrac{1}{2}) - \mathrm{F}(0)) + (\mathrm{F}(2) - \mathrm{F}(\tfrac{1}{2})) \\ &= \mathrm{F}(2) + \mathrm{F}(0) - 2\mathrm{F}(\tfrac{1}{2}) \\ &= (\ln 9 + 1) + 3 - 2\left(\ln\frac{9}{4} + 2\right) \quad \text{M1} \\ &= \ln 9 + \ln\frac{16}{81} = \ln\frac{16}{9}. \quad \text{A1}\end{aligned}$$

9 **(a)** The terms $a, a+2d, a+14d$ in the arithmetic series form a geometric progression. Hence,

$$\begin{aligned}\frac{a+2d}{a} &= \frac{a+14d}{a+2d} \quad \text{M1} \\ (a+2d)^2 &= a(a+14d) \\ a^2 + 4ad + 4d^2 &= a^2 + 14ad \\ 10ad - 4d^2 &= 0 \\ 2d(5a - 2d) &= 0 \quad \text{M1} \\ d &= \frac{5}{2}a, \quad \text{A1}\end{aligned}$$

where we rejected $d \neq 0$ by the question.

Remark: Strictly speaking, $\{a_n\}$ is called a *sequence* while the partial sums $S_n = a_1 + \cdots + a_n$ is called a *series*.

(b) **(i)** Since $0 < \theta < \frac{1}{2}\pi$, $|-\cos\theta| < 1$, and the geometric series converges with sum to infinity

$$\begin{aligned}\frac{\sin\theta}{1-(-\cos\theta)} &= \frac{\sin\theta}{1+\cos\theta} \quad \text{M1}\\ &= \frac{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\cos^2\frac{1}{2}\theta} \quad \text{M1}\\ &= \frac{\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}\\ &= \tan\frac{1}{2}\theta, \quad k = \frac{1}{2}. \quad \text{A1}\end{aligned}$$

(ii) For $\theta = \frac{1}{3}\pi$, $\sin\theta = \frac{1}{2}\sqrt{3}$ and $\cos\theta = \frac{1}{2}$. Thus, the sum of the first 7 terms is given by

$$\begin{aligned}S_7 &= \frac{\sin\theta\cdot\left(1-(-\cos\theta)^7\right)}{1-(-\cos\theta)}\\ &= \frac{\frac{1}{2}\sqrt{3}\cdot\left(1-\left(-\frac{1}{2}\right)^7\right)}{1+\frac{1}{2}} \quad \text{M1}\\ &= \frac{\sqrt{3}}{3}\cdot\frac{129}{128} = \frac{43}{128}\sqrt{3}. \quad \text{A1}\end{aligned}$$

10 **(a)** Differentiating,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a - \frac{a+2b}{(x-1)^2}. \quad \text{M1}$$

At the stationary points, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$:

$$\begin{aligned}a - \frac{a+2b}{(x-1)^2} &= 0\\ (x-1)^2 &= \frac{a+2b}{a}. \quad \text{M1}\end{aligned}$$

Since there are no stationary points, the equation has no solutions. Thus,

$$\begin{aligned}\frac{a+2b}{a} &< 0\\ a+2b &< 0. \quad \text{A1}\end{aligned}$$

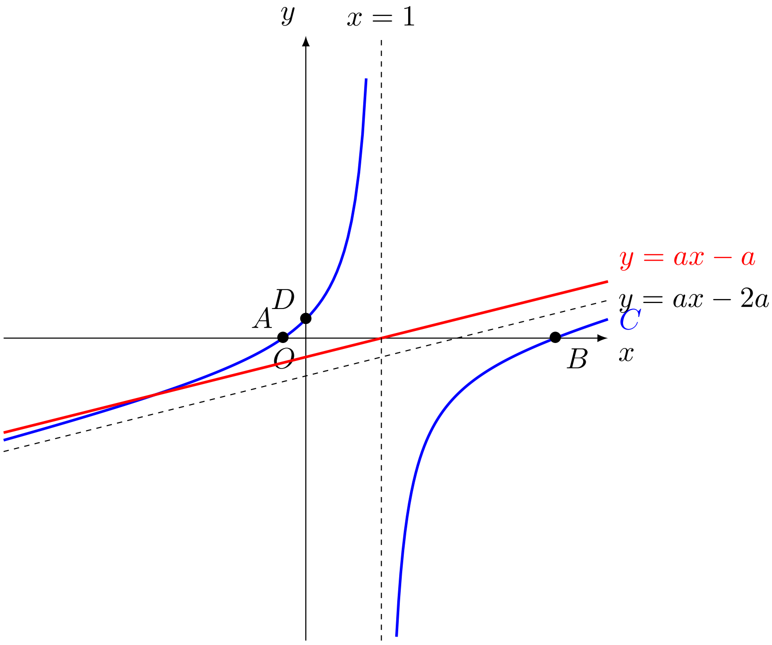
(b) Setting $b = -2a$,

$$y = ax - 2a - \frac{3a}{x-1}.$$

Thus it has asymptotes $y = ax - 2a$ and $x = 1$, as well as y-intercept $(0, a)$. M1
To find x-intercepts, set $y = 0$:

$$\begin{aligned}ax - 2a - \frac{3a}{x-1} &= 0\\ \frac{3a}{x-1} &= a(x-2)\\ (x-2)(x-1) &= 3\\ x^2 - 3x - 1 &= 0.\end{aligned}$$

Using G.C., $x = 3.3027, -0.30277$. M1
Denote $A(-0.303, 0)$, $B(3.30, 0)$, and $D(0, a)$.



Graph: M1 M1

(c) Graph: M1

(d) Setting $a = 1$, the curve and the line have equations

$$y = x - 2 - \frac{3}{x-1}, \quad y = x - 1$$

respectively. Using G.C., they intersect at $(-2, -3)$. M1
Hence, $x \leqslant -2$ or $x > -1$. A1

11 **(a)** By calculating the length of PQ,

$$\begin{aligned}(1136-200)^2 + (92-20)^2 + (p-(-15))^2 &= 939^2 \quad \text{M1}\\ (p+15)^2 &= 441.\end{aligned}$$

Using G.C., $p = -15 \pm 21 = 6, -36$.
Since P lies below Q, we reject $p = 6$ M1 and conclude that $p = -36$. A1

(b) A normal vector to the plane is given by

$$\begin{aligned}\begin{pmatrix}500-400\\200-600\\-70-(-20)\end{pmatrix} \times \begin{pmatrix}600-400\\-340-600\\-50-(-20)\end{pmatrix} &= \begin{pmatrix}100\\-400\\-50\end{pmatrix} \times \begin{pmatrix}200\\-940\\-30\end{pmatrix} \quad \text{M1}\\ &= 100\begin{pmatrix}10\\-40\\-5\end{pmatrix} \times \begin{pmatrix}20\\-94\\-3\end{pmatrix} \parallel \begin{pmatrix}5\\1\\2\end{pmatrix}. \quad \text{M1}\end{aligned}$$

A point on the plane is given by $(400, 600, -20)$. Thus the plane has equation

$$\begin{aligned}\mathbf{r} \cdot \begin{pmatrix}5\\1\\2\end{pmatrix} &= \begin{pmatrix}400\\600\\-20\end{pmatrix} \cdot \begin{pmatrix}5\\1\\2\end{pmatrix} \quad \text{M1}\\ 5x + y + 2z &= 2560. \quad \text{A1}\end{aligned}$$

(c) Since $p = -36$, the line PQ has direction vector

$$\begin{pmatrix}1136\\92\\-36\end{pmatrix} - \begin{pmatrix}200\\20\\-15\end{pmatrix} = \begin{pmatrix}936\\72\\-21\end{pmatrix} \parallel \begin{pmatrix}312\\24\\-7\end{pmatrix} \quad \text{M1}$$

and hence equation

$$PQ : \mathbf{r} = \begin{pmatrix}200\\20\\-15\end{pmatrix} + \lambda \begin{pmatrix}312\\24\\-7\end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

At the desired point of intersection,

$$\begin{aligned}\left(\begin{pmatrix}200\\20\\-15\end{pmatrix} + \lambda \begin{pmatrix}312\\24\\-7\end{pmatrix}\right) \cdot \begin{pmatrix}5\\1\\2\end{pmatrix} &= 2560 \quad \text{M1}\\ \begin{pmatrix}200\\20\\-15\end{pmatrix} \cdot \begin{pmatrix}5\\1\\2\end{pmatrix} + \lambda \begin{pmatrix}312\\24\\-7\end{pmatrix} \cdot \begin{pmatrix}5\\1\\2\end{pmatrix} &= 2560\\ 990 + 1570\lambda &= 2560\\ \lambda &= 1. \quad \text{M1}\end{aligned}$$

Hence,

$$\mathbf{r} = \begin{pmatrix}200\\20\\-15\end{pmatrix} + 1 \cdot \begin{pmatrix}312\\24\\-7\end{pmatrix} = \begin{pmatrix}512\\44\\-22\end{pmatrix}.$$

The point of intersection has coordinates $(512, 44, -22)$. A1

(d) Since the horizontal has normal vector $\begin{pmatrix}0\\0\\1\end{pmatrix}$, the pipeline makes an acute angle θ with the horizontal, where

$$\begin{aligned}\left|\begin{pmatrix}312\\24\\-7\end{pmatrix} \cdot \begin{pmatrix}0\\0\\1\end{pmatrix}\right| &= \left|\begin{pmatrix}312\\24\\-7\end{pmatrix}\right|\left|\begin{pmatrix}0\\0\\1\end{pmatrix}\right| \cos(90^\circ - \theta) \quad \text{M1}\\ 7 &= 313\cos(90^\circ - \theta)\\ 90^\circ - \theta &= 88.718^\circ\\ \theta &= 1.2814^\circ \approx 1.3^\circ. \quad \text{A1}\end{aligned}$$

12 **(a)** The differential equation is given by

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP. \quad \text{M1}$$

Thus,

$$\begin{aligned}\int \frac{1}{P}\,\mathrm{d}P &= \int k\,\mathrm{d}t \quad \text{M1}\\ \ln|P| &= kt + C\\ P &= \pm \mathrm{e}^C \mathrm{e}^{kt}\\ &= A\mathrm{e}^{kt}. \quad \text{M1}\end{aligned}$$

Since $P(0) = 50$, $A = 50$, and

$$P = 50\mathrm{e}^{kt}. \quad \text{M1}$$

Since $P(10) = 100$,

$$100 = 50\mathrm{e}^{10k}$$
$$\mathrm{e}^k = 2^{\frac{1}{10}}.$$

Therefore, $P = 50 \cdot 2^{\frac{t}{10}}$. A1

(b) We have

$$\int \frac{1}{P(500-P)}\,\mathrm{d}P = \int \lambda\,\mathrm{d}t.$$

We simplify the left hand side as follows:

$$\begin{aligned}\frac{1}{P(500-P)} &= \frac{1}{500P - P^2}\\ &= \frac{1}{250^2 - (P-250)^2}. \quad \text{M1}\end{aligned}$$

Hence, using MF26,

$$\begin{aligned}\int \frac{1}{P(500-P)}\,\mathrm{d}P &= \int \lambda\,\mathrm{d}t\\ \int \frac{1}{250^2-(P-250)^2}\,\mathrm{d}P &= \int \lambda\,\mathrm{d}t\\ \frac{1}{2\cdot 250}\ln\left|\frac{250+(P-250)}{250-(P-250)}\right| &= \lambda t + C \quad \text{M1}\\ \frac{P}{500-P} &= \pm\mathrm{e}^{500C}\mathrm{e}^{500\lambda t} = A\mathrm{e}^{500\lambda t}\\ P &= 500A\mathrm{e}^{500\lambda t} - PA\mathrm{e}^{500\lambda t}\\ P + PA\mathrm{e}^{500\lambda t} &= 500A\mathrm{e}^{500\lambda t}\\ P &= \frac{500A\mathrm{e}^{500\lambda t}}{1+A\mathrm{e}^{500\lambda t}} = \frac{500A}{A+\mathrm{e}^{-500\lambda t}}. \quad \text{M1}\end{aligned}$$

Since $P(0) = 50$,

$$50 = \frac{500A}{1+A} \quad\Rightarrow\quad \frac{10A}{1+A} = 1 \quad\Rightarrow\quad A = \frac{1}{9}, \quad \text{M1}$$

thus

$$P = \frac{500\cdot\frac{1}{9}}{\frac{1}{9}+\mathrm{e}^{-500\lambda t}} = \frac{500}{1+9\mathrm{e}^{-500\lambda t}}.$$

Since $P(10) = 100$,

$$100 = \frac{500}{1+9\mathrm{e}^{-500\lambda\cdot 10}}$$
$$\mathrm{e}^{-500\lambda} = \left(\frac{4}{9}\right)^{\frac{1}{10}}. \quad \text{M1}$$

Hence,

$$P = \frac{500}{1+9\left(\frac{4}{9}\right)^{\frac{t}{10}}}. \quad \text{A1}$$

(c) In the long run, $t \to \infty$, thus $\left(\frac{4}{9}\right)^{\frac{t}{10}} \to 0$, and $P \to 500$. B1
This is an improvement as it accounts for a slowdown in growth due to overpopulation, while the first model allows for infinite population, which is absurd in our finite world. B1