H2 MATHEMATICS

Paper 1

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3 hours

Proposed Solutions

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1 Differentiating with respect to x on both sides,

$$\frac{1}{y}\cdot\frac{\mathrm{d}y}{\mathrm{d}x} = 2(11-5x)\cdot(-5). \quad \text{M1} \quad \text{M1}$$

Setting $x = 2$,

$$\ln(y) = (11-5\cdot 2)^2 = 1 \quad \Rightarrow \quad y = e. \quad \text{M1}$$

Substituting,

$$\frac{1}{e}\frac{\mathrm{d}y}{\mathrm{d}x} = 2(11-5\cdot 2)\cdot(-5) = -10 \quad \Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = -10e. \quad \text{M1}$$

Hence the tangent has equation

$$y - e = -10e(x-2) \quad \iff \quad y = -10ex + 21e. \quad \text{A1}$$

2 **(a)** Let $u_n = an^3 + bn^2 + cn + d$ for some real constants a, b, c, d. Substituting $n = 1, 2, 3, 4$ respectively,

$$\begin{aligned} a + b + c + d &= 10 \\ 8a + 4b + 2c + d &= 61 \\ 27a + 9b + 3c + d &= 206 \\ 64a + 16b + 4c + d &= 469. \quad \text{M1} \end{aligned}$$

Using G.C.,

$$(a, b, c, d) = (4, 23, -46, 29). \quad \text{M1}$$

Hence,

$$u_n = 4n^3 + 23n^2 - 46n + 29. \quad \text{A1}$$

(b) We want to solve

$$u_n > 25\,000 \quad \iff \quad v_n := u_n - 25\,000 > 0.$$

Using G.C., $v_{16} = -3435 < 0$ and $v_{17} = 546 > 0$. M1 Hence, $n \geqslant 17$. A1

3 **(a)** It is known that $\mathbf{a}\times\mathbf{b} \perp \mathbf{a}$, thus $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} = 0$. Similarly, $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} = 0$. Since $(\mathbf{a}\times\mathbf{b}+\mathbf{a}) \perp (\mathbf{a}\times\mathbf{b}+\mathbf{b})$,

$$\begin{aligned} 0 &= (\mathbf{a}\times\mathbf{b}+\mathbf{a})\cdot(\mathbf{a}\times\mathbf{b}+\mathbf{b}) \\ &= |\mathbf{a}\times\mathbf{b}|^2 + \mathbf{a}\cdot(\mathbf{a}\times\mathbf{b}) + \mathbf{b}\cdot(\mathbf{a}\times\mathbf{b}) + \mathbf{a}\cdot\mathbf{b} \quad \text{M1} \\ &= |\mathbf{a}\times\mathbf{b}|^2 + 0 + 0 + (-1) \quad \text{M1} \\ &= |\mathbf{a}\times\mathbf{b}|^2 - 1. \end{aligned}$$

Since $|\mathbf{a}\times\mathbf{b}| > 0$,

$$|\mathbf{a}\times\mathbf{b}|^2 = 1 \quad \Rightarrow \quad |\mathbf{a}\times\mathbf{b}| = 1. \quad \text{M1}$$

(b) Let θ denote the angle between the direction of $\mathbf{a}$ and the direction of $\mathbf{b}$. Since $|\mathbf{a}\times\mathbf{b}| = 1 = -(\mathbf{a}\cdot\mathbf{b})$,

$$\begin{aligned}|\mathbf{a}||\mathbf{b}|\sin(\theta) &= -|\mathbf{a}||\mathbf{b}|\cos(\theta) \quad \text{M1}\\ \tan(\theta) &= -1 \quad \text{M1}\\ \theta &= 180^\circ - 45^\circ\\ &= 135^\circ. \quad \text{A1}\end{aligned}$$

4 (a) Write

$$\cos(px)\cos(qx) = \frac{1}{2}\cos((p+q)x) + \frac{1}{2}\cos((p-q)x).$$

Since $p \neq \pm q$, $p \pm q \neq 0$. Hence,

$$\begin{aligned}\int (\cos(px)\cos(qx))\,\mathrm{d}x &= \frac{1}{2}\int (\cos((p+q)x) + \cos((p-q)x))\,\mathrm{d}x \quad \text{M1}\\ &= \frac{1}{2}\left[\frac{\sin((p+q)x)}{p+q} + \frac{\sin((p-q)x)}{p-q}\right] + C. \quad \text{A1}\end{aligned}$$

(b) Integrating by parts,

$$\begin{aligned}\int x\cos(nx)\,\mathrm{d}x &= \underbrace{\frac{\sin(nx)}{n}}_{\text{I}}\underbrace{x}_{\text{S}} - \int \underbrace{\frac{\sin(nx)}{n}}_{\text{I}}\underbrace{1}_{\text{D}}\,\mathrm{d}x \quad \text{M1}\\ &= \frac{x\sin(nx)}{n} - \frac{1}{n}\int \sin(nx)\,\mathrm{d}x\\ &= \frac{x\sin(nx)}{n} - \frac{1}{n^2}(-\cos(nx)) + C \quad \text{M1}\\ &= \frac{x\sin(nx)}{n} + \frac{\cos(nx)}{n^2} + C. \quad \text{A1}\end{aligned}$$

(c) From **(b)**, since $\sin(n\pi) = 0$ for any $n \in \mathbb{Z}^+$ and $\cos(n\pi) = (-1)^n$,

$$\begin{aligned}\int_0^\pi x\cos(nx)\,\mathrm{d}x &= \left[\frac{x\sin(nx)}{n} + \frac{\cos(nx)}{n^2}\right]_0^\pi\\ &= \left(\frac{\pi\sin(n\pi)}{n} + \frac{\cos(n\pi)}{n^2}\right) - \left(\frac{0\sin(n\cdot 0)}{n} + \frac{\cos(n\cdot 0)}{n^2}\right)\\ &= \frac{(-1)^n}{n^2} - \frac{1}{n^2}\\ &= \begin{cases}\frac{1}{n^2} - \frac{1}{n^2} & \text{if } n \text{ is even}\\ -\frac{1}{n^2} - \frac{1}{n^2} & \text{if } n \text{ is odd}\end{cases}\\ &= \begin{cases}0 & \text{if } n \text{ is even}\\ -\frac{2}{n^2} & \text{if } n \text{ is odd.}\end{cases} \quad \text{A1}\end{aligned}$$

That is, $k = 0$ if n is even and $k = -2$ if n is odd. A1

(d) From **(b)**,

$$\begin{aligned}\int_0^{\frac{\pi}{2}} |x\cos(2x)|\,\mathrm{d}x &= \int_0^{\frac{\pi}{4}} |x\cos(2x)|\,\mathrm{d}x + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} |x\cos(2x)|\,\mathrm{d}x\\ &= \int_0^{\frac{\pi}{4}} x\cos(2x)\,\mathrm{d}x - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\cos(2x)\,\mathrm{d}x \quad \text{M1}\\ &= \left(\frac{\pi}{8} - \frac{1}{4}\right) - \left(-\frac{1}{4} - \frac{\pi}{8}\right) = \frac{1}{4}\pi. \quad \text{M1} \quad \text{A1}\end{aligned}$$

5 **(a)** By logarithm properties,

$$\ln\left[\frac{(r-1)(r+1)}{r^2}\right] = \ln(r-1) - 2\ln(r) + \ln(r+1).$$

Summing on both sides,

$$\begin{aligned}
&\sum_{r=2}^{n} \ln\left[\frac{(r-1)(r+1)}{r^2}\right] \\
&= \sum_{r=2}^{n} (\ln(r-1) - 2\ln(r) + \ln(r+1)) \quad \text{M1} \\
&= \ln(1) + \ln(2) + \ln(3) + \cdots + \ln(n-3) + \ln(n-2) + \ln(n-1) \\
&\quad - 2\ln(2) - 2\ln(3) - 2\ln(4) - \cdots - 2\ln(n-2) - 2\ln(n-1) - 2\ln(n) \\
&\quad + \ln(3) + \ln(4) + \ln(5) + \cdots + \ln(n-1) + \ln(n) + \ln(n+1) \quad \text{M1} \\
&= \ln(1) - \ln(2) - \ln(n) + \ln(n+1) \\
&= \ln\left(\frac{n+1}{n}\right) - \ln(2). \quad \text{M1}
\end{aligned}$$

(b) By algebruh, taking $n \to \infty$,

$$\sum_{r=2}^{n} \ln\left[\frac{(r-1)(r+1)}{r^2}\right] = \ln\left(1 + \frac{1}{n}\right) - \ln(2) \to \ln(1+0) - \ln(2) = -\ln(2). \quad \text{M1}$$

Hence, the infinite series converges to its sum to infinity $-\ln(2)$. A1

(c) By using $r = 2$ as the starting point,

$$\begin{aligned}
\sum_{r=10}^{20} \ln\left[\frac{(r-1)(r+1)}{r^2}\right] &= \sum_{r=2}^{20} \ln\left[\frac{(r-1)(r+1)}{r^2}\right] - \sum_{r=2}^{9} \ln\left[\frac{(r-1)(r+1)}{r^2}\right] \\
&= \left(\ln\left(\frac{21}{20}\right) - \ln(2)\right) - \left(\ln\left(\frac{10}{9}\right) - \ln(2)\right) \quad \text{M1} \\
&= \ln\left(\frac{21}{20} \cdot \frac{9}{10}\right) = \ln\left(\frac{189}{200}\right), \quad \text{A1}
\end{aligned}$$

where $(a, b) = (189, 200)$.

6 **(a)** By the double angle formulae,

$$\begin{aligned}
\cos^4(\theta) &= \left(\cos^2(\theta)\right)^2 \\
&= \left(\frac{1}{2}(1 + \cos(2\theta))\right)^2 \quad \text{M1} \\
&= \frac{1}{4}(1 + \cos(2\theta))^2 \\
&= \frac{1}{4}(1 + 2\cos(2\theta) + \cos^2(2\theta)) \\
&= \frac{1}{4}\left(1 + 2\cos(2\theta) + \frac{1}{2}(1 + \cos(4\theta))\right) \\
&= \frac{1}{4}\left(\frac{3}{2} + 2\cos(2\theta) + \frac{1}{2}\cos(4\theta)\right) \quad \text{M1} \\
&= \frac{1}{8}(3 + 4\cos(2\theta) + \cos(4\theta)),
\end{aligned}$$

which equals $\frac{1}{8}(\cos(4\theta) + 4\cos(2\theta) + 3)$.

(b) The required volume V is given by

$$\frac{V}{\pi} = \int_{1.5}^{3} y^2 \, \mathrm{d}x = \int_{1.5}^{3} (9 - x^2)^{\frac{3}{2}} \, \mathrm{d}x. \quad \text{M1}$$

Using the substitution $x = 3\sin(\theta)$,

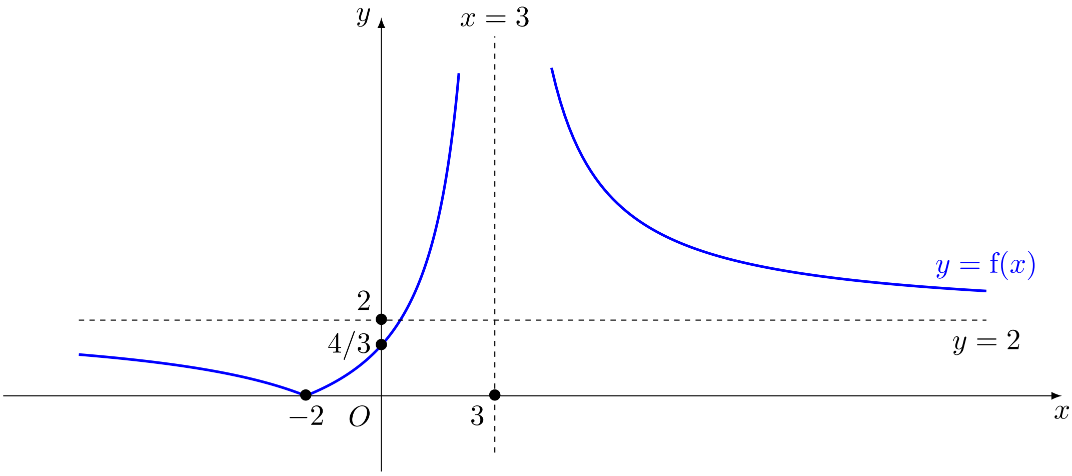
$$\begin{aligned} 9 - x^2 &= 9 - 9\sin^2(\theta) \\ &= 9\cos^2(\theta) \\ \left(9 - x^2\right)^{\frac{3}{2}} &= \left(9\cos^2(\theta)\right)^{\frac{3}{2}} \\ &= 27\cos^3(\theta), \\ \frac{\mathrm{d}x}{\mathrm{d}\theta} &= 3\cos(\theta). \quad \text{M1} \end{aligned}$$

When $x = 1.5$, $\theta = \frac{1}{6}\pi$. When $x = 3$, $\theta = \frac{1}{2}\pi$. Making the substitutions,

$$\begin{aligned} \int_{1.5}^{3} (9 - x^2)^{\frac{3}{2}} \, \mathrm{d}x &= \int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi} \left(81\cos^4(\theta)\right) \mathrm{d}\theta \quad \text{M1} \\ &= \frac{81}{8} \int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi} \left(\cos(4\theta) + 4\cos(2\theta) + 3\right) \mathrm{d}\theta \\ &= \frac{81}{8} \left[\frac{1}{4}\sin(4\theta) + 2\sin(2\theta) + 3\theta \right]_{\frac{1}{6}\pi}^{\frac{1}{2}\pi} \quad \text{M1} \\ &= \frac{81}{8} \left(\left(0 + 0 + \frac{3}{2}\pi \right) - \left(\frac{1}{4} \cdot \frac{\sqrt{3}}{2} + 2 \cdot \frac{\sqrt{3}}{2} + \frac{1}{2}\pi \right) \right) \quad \text{M1} \\ &= \frac{81}{8} \left(\pi - \frac{9}{8}\sqrt{3} \right). \end{aligned}$$

Hence, $V = \frac{81}{8}\pi\left(\pi - \frac{9}{8}\sqrt{3}\right)$ units^3. A1

7 **(a)** We sketch as follows.



Curve: M1 , asymptotes: M1 , axial-intercepts: M1

(b) By the graph, $\mathrm{R_f} = [0, \infty)$. B1

(c) Since $\mathrm{R_f} = [0, \infty) \not\subseteq \mathbb{R} \backslash \{3\} = \mathrm{D_f}$, the composite function f^2 does not exist. B1

(d) For f^{-1} to exist, f must be 1-1, thus $a = -2$. B1

(e) For $x \leqslant -2$, $\dfrac{2x+4}{3-x} \leqslant 0$. Hence,

$$f(x) = \left|\frac{2x+4}{3-x}\right| = -\frac{2x+4}{3-x}. \quad \text{M1}$$

Let $y = \mathrm{f}^{-1}(x) \iff x = \mathrm{f}(y)$. Then

$$\begin{aligned} x &= -\frac{2y+4}{3-y} \quad \text{M1} \\ (3-y)x &= -(2y+4) \\ xy - 3x &= 2y + 4 \\ (x-2)y &= 3x + 4 \\ y &= \frac{3x+4}{x-2} \\ \mathrm{f}^{-1}(x) &= \frac{3x+4}{x-2}. \quad \text{A1} \end{aligned}$$

Furthermore, with the restricted domain $(-\infty, -2]$, $\mathrm{D_{f^{-1}}} = \mathrm{R_f} = [0, 2)$. A1

8 **(a)** **(i)** We have

$$|z| = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2, \quad \arg(z) = \pi - \frac{1}{3}\pi = \frac{2}{3}\pi \quad \Rightarrow \quad z = 2e^{\mathrm{i}\left(\frac{2}{3}\pi\right)}. \quad \text{M1} \quad \text{A1}$$

(ii) We note that

$$\frac{z^n}{iz^*} = \frac{2^n \cdot e^{\mathrm{i}\left(\frac{2}{3}n\right)\pi}}{e^{\mathrm{i}\left(\frac{1}{2}\pi\right)}2e^{-\mathrm{i}\left(\frac{2}{3}\pi\right)}} = 2^n e^{\mathrm{i}\left(\frac{2}{3}n+\frac{2}{3}-\frac{1}{2}\right)\pi} = 2^n e^{\mathrm{i}\left(\frac{2}{3}n+\frac{1}{6}\right)\pi}.$$

For $\dfrac{z^n}{iz^*}$ to be purely imaginary,

$$\begin{aligned} \left(\frac{2}{3}n + \frac{1}{6}\right)\pi &= (2k+1) \cdot \frac{\pi}{2} \quad \text{M1} \\ \frac{2}{3}n + \frac{1}{6} &= k + \frac{1}{2} \\ n &= \frac{3k+1}{2}. \quad \text{M1} \end{aligned}$$

For n to be the smallest positive integer, we require $k = 1 \Rightarrow n = 2$. A1

(b) From the first equation,

$$v = \frac{1-|w|}{2} \in \mathbb{R}. \quad \text{M1}$$

Furthermore,

$$|w|^2 = (1-2v)^2$$

From the second equation,

$$\begin{aligned} -\mathrm{i}w &= -3 - 3v + 4\mathrm{i} \\ w &= (-3-3v)\mathrm{i} + 4(-1) \\ &= -4 - 3(v+1)\mathrm{i} \quad \text{M1} \\ |w|^2 &= 16 + 9(v+1)^2. \end{aligned}$$

Hence,

$$\begin{aligned} 16 + 9(v+1)^2 &= (1-2v)^2 \quad \text{M1} \\ 16 + 9(v^2 + 2v + 1) &= 1 - 4v + 4v^2 \\ 9v^2 + 18v + 25 &= 1 - 4v + 4v^2 \\ 5v^2 + 23v + 24 &= 0 \\ (5v+12)(v+2) &= 0. \end{aligned}$$

Hence, $v = -2$ or $v = -\frac{12}{5}$. A1 Substituting,

$$w = -4 + 3\mathrm{i} \quad \text{or} \quad w = -4 + \frac{21}{5}\mathrm{i}.$$

Therefore, $(v, w) = (-2, -4+3\mathrm{i})$ or $(v, w) = (-\frac{12}{5}, -4 + \frac{21}{5}\mathrm{i})$. A1

9 The equations of l_1, l_2 are given in vector form by

$$l_1 : \mathbf{r} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ a \end{pmatrix}, \quad \lambda \in \mathbb{R},$$

$$l_2 : \mathbf{r} = \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \quad \mu \in \mathbb{R}. \quad \text{M1}$$

(a) At the point of intersection,

$$\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ a \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}. \quad \text{M1}$$

Equating components,

$$\begin{aligned} 2\lambda - 3\mu &= 1 \\ \lambda - 2\mu &= 0 \\ a\lambda - \mu &= 3. \end{aligned}$$

Since the lines intersect, this system of equations has solutions. Solving the first two equations, $(\lambda, \mu) = (2, 1)$. M1 Substituting into the last equation,

$$a \cdot 2 - 1 = 3 \quad \Rightarrow \quad a = 2. \quad \text{A1}$$

Substituting $\mu = 1$, B has position vector

$$\overrightarrow{OB} = \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix} + 1 \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}.$$

Hence, $B(1, 3, 6)$. A1

(b) Since π_1 is perpendicular to l_2, it has normal vector parallel to $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$. Since π_1 contains $A(-3, 1, 2)$, it has scalar product equation

$$\pi_1 : \mathbf{r} \cdot \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = -5.$$

(i) By drawing a diagram, the shortest distance p from B to π_1 is the length of projection of

$$\overrightarrow{BA} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} - \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix}$$

to the normal vector $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$, computed via

$$\begin{aligned} p &= \left| \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix} \cdot \left(\frac{1}{\sqrt{3^2 + 2^2 + 1^2}} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right) \right| \quad \text{M1} \\ &= \frac{1}{\sqrt{14}} (12 + 4 + 4) \quad \text{M1} \\ &= \frac{20}{\sqrt{14}} = \frac{10}{7}\sqrt{14} \text{ units.} \quad \text{A1} \end{aligned}$$

(ii) We note that $\overrightarrow{AB}$ is parallel to the direction vector of l_1. By drawing a diagram, the required angle θ is obtained via trigonometry:

$$\sin(\theta) = \frac{\text{(opposite)}}{\text{(hypotenuse)}} = \frac{\frac{10}{7}\sqrt{14}}{\sqrt{4^2+2^2+4^2}} = \frac{10}{7} \cdot \frac{\sqrt{14}}{6} = \frac{5\sqrt{14}}{21}. \quad \text{M1}$$

Using G.C., $\theta = 62.982° \approx 63.0°$. A1

(c) Since π_2 contains l_1, it contains $A(-3, 1, 2)$ and is parallel to $\begin{pmatrix}2\\1\\2\end{pmatrix}$. Since π_2 is perpendicular to π_1, it is parallel to $\begin{pmatrix}3\\2\\1\end{pmatrix}$. Hence π_1 has a normal vector parallel to

$$\begin{pmatrix}2\\1\\2\end{pmatrix} \times \begin{pmatrix}3\\2\\1\end{pmatrix} = \begin{pmatrix}-3\\4\\1\end{pmatrix}. \quad \text{M1} \quad \text{M1}$$

Thus,

$$\pi_2 : \mathbf{r} \cdot \begin{pmatrix}-3\\4\\1\end{pmatrix} = \begin{pmatrix}-3\\1\\2\end{pmatrix} \cdot \begin{pmatrix}-3\\4\\1\end{pmatrix} = 15 \quad \Longleftrightarrow \quad \pi_2 : -3x + 4y + z = 15. \quad \text{A1}$$

10 **(a)** By the question,

$$\begin{aligned}\frac{\mathrm{d}M}{\mathrm{d}t} &= k\left(\left.\frac{\mathrm{d}M}{\mathrm{d}t}\right|_{\text{in}} - \left.\frac{\mathrm{d}M}{\mathrm{d}t}\right|_{\text{out}}\right) \\ &= k(C - 30M). \quad \text{B1}\end{aligned}$$

(b) For his mass to remain at 110 kg, $\frac{\mathrm{d}M}{\mathrm{d}t} = 0$:

$$C = 30M = 30 \cdot 110 = 3300 \text{ Calories per day.} \quad \text{B1}$$

(c) By the question, $C = 0.8 \cdot 3300 = 2640$. From **(a)**,

$$\begin{aligned}\int \frac{-30}{C - 30M} \, \mathrm{d}M &= \int -30k \, \mathrm{d}t \quad \text{M1} \\ \ln|C - 30M| &= -30kt + C_1 \\ C - 30M &= \pm e^{C_1} e^{-30kt} = Ae^{-30kt} \quad (A = \pm e^{C_1}) \\ M &= \frac{C - Ae^{-30kt}}{30} \\ &= \frac{C - Ae^{-30kt}}{30}. \quad \text{M1}\end{aligned}$$

Since $M(0) = 110$,

$$110 = \frac{C - A}{30} \quad \Rightarrow \quad A = C - 3300 = 2640 - 3300 = -660. \quad \text{M1}$$

Hence,

$$M = \frac{C + 660e^{-30kt}}{30} = \frac{2640 + 660e^{-30kt}}{30} = 88 + 22e^{-30kt}. \quad \text{A1}$$

(d) By the question $M(75) = 100$. Hence,

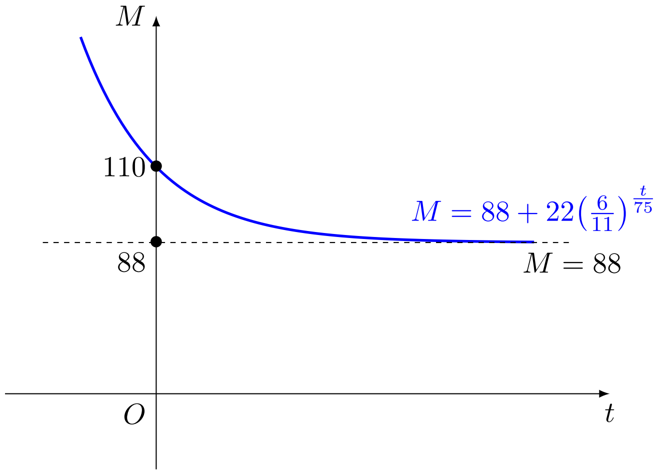
$$88 + 22e^{-30k(75)} = 100 \quad \Rightarrow \quad e^{30k(75)} = \frac{6}{11} \quad \Rightarrow \quad e^{-30k} = \left(\frac{6}{11}\right)^{\frac{1}{75}}. \quad \text{M1}$$

We want to find the smallest integer t such that $M(75+t) < 96$:

$$88 + 22e^{-30k(75+t)} < 96 \quad \text{M1}$$
$$88 + 22e^{-30k(75)}e^{-30kt} < 96$$
$$88 + 22 \cdot \frac{6}{11} \cdot \left(\frac{6}{11}\right)^{\frac{t}{75}} < 96. \quad \text{M1}$$

Using G.C., $t > 50.17$. Thus, the smallest integer t is 51 days. A1

(e) **(i)** We sketch as follows. A1



As $t \to \infty$, $M \to 88 + 22 \cdot 0 = 88 > 80$, and Andrew cannot achieve a mass of 80 kg. B1

(ii) We observe from **(c)** that

$$M = \frac{C}{30} + Ke^{-30kt}$$

for some constant $K > 0$. Since $M \to C/30$ as $t \to \infty$, we require

$$0 < \frac{C}{30} < 80 \quad \Rightarrow \quad 0 < C < 2400 \text{ Calories per day.} \quad \text{A1}$$

11 **(a)** Wei's monthly deposit follows an arithmetic progression with first term a and common difference 50. The total amount S_n after n deposits is given by

$$S_n = \frac{n}{2}(a + (a + 50(n-1)) = \frac{n}{2}(2a - 50 + 50n).$$

On 31 December 2023, $n = 36$. Thus we want to find a such that

$$S_{36} > 50\,000$$
$$\frac{36}{2}(2a - 50 + 50 \cdot 36) > 50\,000 \quad \text{M1}$$

Using G.C., $a > 513.8888$. Thus the smallest value of a is $\$a = \$513.89$ (nearest cent). A1

(b) Denote the initial loan by $P = 400\,000$, the interest rate by $r = 0.001$, the multiplier by $M := 1+r = 1.001$, and the monthly repayment by $c = -x$. We tabulate the outstanding loans as follows.

Month	Start	End
1	PM	$PM + c$
2	$PM^2 + cM$	$PM^2 + cM + c$
3	$PM^3 + cM^2 + cM$	$PM^3 + cM^2 + cM + c$

Repeating the pattern, the outstanding loan T_n at the end of n months is

$$\begin{aligned} T_n &= PM^n + cM^{n-1} + cM^{n-2} + \cdots + cM + c \quad \text{M1} \\ &= PM^n + c \cdot (1 + M + \cdots + M^{n-2} + M^{n-1}) \\ &= PM^n + c \cdot \frac{1 - M^n}{1 - M} \quad \text{M1} \\ &= PM^n + c \cdot \frac{M^n - 1}{M - 1} \\ &= 400\,000 \times 1.001^n + (-x) \cdot \frac{1.001^n - 1}{1.001 - 1} \\ &= 400\,000 \times 1.001^n - 1000x(1.001^n - 1). \quad \text{M1} \end{aligned}$$

(c) **(i)** By the question, $T_{360} = 0$:

$$400\,000 \times 1.001^{360} - 1000x(1.001^{360} - 1) = 0 \quad \text{M1}$$

Using G.C., the monthly repayment is $\$x = \$1323.6347 \approx \$1323.64$ (round up to the nearest cent). A1

(ii) The total amount paid is $360x$. Hence, the total interest paid is

$$360x - 400\,000 = 76\,510.40 \approx \$76\,500. \quad \text{A1}$$

(d) **(i)** Set $x = 1600$ in the result in **(b)**. We want to solve

$$400\,000 \times 1.001^k - 1000 \cdot 1600 \cdot (1.001^k - 1) < 1600 \quad \text{M1}$$

Using G.C., $k > 286.82$. Thus, the least number k is 287. A1 The outstanding amount y is thus given by

$$y = 400\,000 \times 1.001^{287} - (1000 \cdot 1600 \cdot (1.001^{287} - 1)) = 1320.2179 \approx \$1320.22. \quad \text{(nearest cent)}$$

M1 A1

(ii) The cost savings in **(d)(i)** compared to **(c)(i)** is computed by

$$360 \cdot 1323.6347 - (287 \cdot 1600 + 1320.2179) = 15\,988.2741 \approx \$15\,990. \quad \text{(4sf)} \quad \text{A1}$$