H2 MATHEMATICS — 9758/01

Paper 1 — October/November 2024

3 hours

Proposed Solutions

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1 Since there is a vertical asymptote at $x = -1/2$,

$$a\left(-\frac{1}{2}\right)^2 + b\left(-\frac{1}{2}\right) + c = 0 \quad \Rightarrow \quad a - 2b + 4c = 0. \quad \text{M1}$$

At the turning point $(-2, -1/9)$,

$$\frac{-2+1}{a(-2)^2 + b(-2) + c} = -\frac{1}{9} \quad \Rightarrow \quad 4a - 2b + c = 9. \quad \text{M1}$$

At a turning point, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. By the quotient rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(ax^2+bx+c)\cdot 1 - (x+1)(2ax+b)}{(ax^2+bx+c)^2}.$$

Substituting $x = -2$ and setting $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$:

$$(4a - 2b + c)\cdot 1 - (-2+1)(-4a+b) = 0 \quad \Rightarrow \quad -8a + 4b + c = 0. \quad \text{M1}$$

Using G.C., $a = 2, b = -1, c = -1$. A1

2 Since $|\mathbf{a} \times \mathbf{b}| = 3$,

$$|\mathbf{a}||\mathbf{b}|\sin\theta = |\mathbf{a} \times \mathbf{b}| = 3, \quad \text{B1}$$

where θ denotes the angle between $\mathbf{a}$ and $\mathbf{b}$. By the question, $|\mathbf{a}| = 1$ and

$$|\mathbf{b}|^2 = \mathbf{b}\cdot\mathbf{b} = 9 \quad \Rightarrow \quad |\mathbf{b}| = 3. \quad \text{M1}$$

Substituting,

$$1 \cdot 3 \cdot \sin\theta = 3 \quad \Rightarrow \quad \sin\theta = 1 \quad \Rightarrow \quad \theta = 90^\circ \quad \Rightarrow \quad \mathbf{a} \perp \mathbf{b}. \quad \text{M1}$$

3 **(a)** The coefficients a, b, and c must be real. B1

Remark: To prove this claim, we first observe that

$$p := w + w^*, \quad q := ww^* = |w|^2$$

are real numbers. By algebruh,

$$\begin{aligned} z^3 + az^2 + bz + c &= (z-3)(z-w)(z-w^*) \\ &= (z-3)(z^2 - (w+w^*) + ww^*) \\ &= (z-3)(z^2 - pz + q) \\ &= z^3 - (p+3)z^2 + (3p+q)z - 3q. \end{aligned}$$

Comparing coefficients, $a = -(p+3), b = 3p+q, c = -3q$ are all real numbers.

(b) Using the **Remark** in part **(a)**,

$$p = (-1+2\mathrm{i}) + (-1-2\mathrm{i}) = -2, \quad q = |1+2\mathrm{i}|^2 = 1^2 + 2^2 = 5. \quad \text{M1} \quad \text{M1}$$

Hence,

$$a = -(-2+3) = -1, \quad b = 3(-2) + 5 = -1, \quad c = -3(5) = -15. \quad \text{A1}$$

4 **(a)** Since the denominator needs to be nonzero, $x \neq -2, 0$. B1 By algebruh,

$$\begin{aligned}\frac{4}{x+2} - \frac{x-3}{x} &= \frac{4x-(x-3)(x+2)}{x(x+2)}\\ &= \frac{4x-(x^2-x-6)}{x(x+2)}\\ &= \frac{-x^2+5x+6}{x(x+2)}\\ &= \frac{-(x^2-5x-6)}{x(x+2)}\\ &= -\frac{(x-6)(x+1)}{x(x+2)}. \quad \text{M1}\end{aligned}$$

Hence,

$$\begin{aligned}\frac{4}{x+2} - \frac{x-3}{x} &\geqslant 0\\ -\frac{(x-6)(x+1)}{x(x+2)} &\geqslant 0\\ \frac{(x-6)(x+1)}{x(x+2)} &\leqslant 0\\ (x+2)(x+1)x(x-6) &\leqslant 0. \quad \text{M1}\end{aligned}$$

which implies $-2 \leqslant x \leqslant -1$ or $0 \leqslant x \leqslant 6$. Thus, the final solution is

$$-2 < x \leqslant -1 \quad \text{or} \quad 0 < x \leqslant 6. \quad \text{A1}$$

(b) Replacing x with $|x|$,

$$-2 < |x| \leqslant -1 \quad \text{or} \quad 0 < |x| \leqslant 6. \quad \text{M1}$$

Since $|x| \geqslant 0$ for any real x, $-2 < |x| \leqslant -1$ yields no solution. Thus,

$$0 < |x| \leqslant 6 \quad \Rightarrow \quad -6 \leqslant x \leqslant 6, \quad x \neq 0. \quad \text{A1}$$

5 **(a)** By the sum of a geometric series,

$$S_\infty = \frac{a}{1-r} = 18. \quad \Rightarrow \quad a = 18(1-r). \quad \text{M1}$$

By the nth term of a geometric series,

$$ar^2 = \frac{8}{3}. \quad \text{M1}$$

Substituting a and performing nursery algebruh,

$$\begin{aligned}18(1-r)r^2 &= \frac{8}{3}\\ 54(1-r)r^2 &= 8\\ 54r^2 - 54r^3 &= 8\\ 54r^2 - 54r^3 - 8 &= 0\\ 27r^3 - 27r^2 + 4 &= 0. \quad \text{A1}\end{aligned}$$

(b) Using G.C., since $r < 0$, $r = -1/3$ and $a = 24$. A1

(c) Define

$$S := \sum_{r=1}^{k} u_r = \frac{ar(1-r^k)}{1-r} = \frac{-8(1-(-1/3)^k)}{1-(-1/3)}.$$

Observe that

$$18 = \sum_{r=0}^{\infty} u_r = 24 + \sum_{r=1}^{k} u_r + \sum_{r=k+1}^{\infty} u_r \quad \Rightarrow \quad \sum_{r=k+1}^{\infty} u_r = -6 - S. \quad \text{M1}$$

Substituting,

$$80(-6-S) = S \quad \Rightarrow \quad \frac{-8(1-(-1/3))^k}{1-(-1/3)} = S = -\frac{160}{27}. \quad \text{M1}$$

Using G.C., $k = 4$. A1

6 **(a)** By the chain rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2 \cdot e^{x^3} \cdot 3x^2 = 6x^2 e^{x^3}. \quad \text{M1}$$

Setting $x = 1$ yields a gradient of

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=1} = 6 \cdot 1^2 \cdot e^{1^3} = 6e. \quad \text{M1}$$

Setting $x = 1$ yields

$$y = 2e^{1^3} + a = 2e + a. \quad \text{M1}$$

Thus, $T(1, 2e + a)$. Thus, the tangent has equation

$$y - (2e + a) = 6e(x - 1). \quad \Rightarrow \quad y = e(6x - 4) + a, \quad \text{M1} \quad \text{A1}$$

where $p = 6, q = -4, r = 1$.

(b) Since the tangent in part **(a)** passes through $O(0, 0)$,

$$0 = e(6 \cdot 0 - 4) + a \quad \Rightarrow \quad a = 4e. \quad \text{A1}$$

(c) Using integration, the required area A is given by

$$\begin{aligned} A &= \int_0^1 (2e^{x^3} + 4e)\,\mathrm{d}x - \frac{1}{2} \cdot 1 \cdot (2e + 4e) \quad \text{M1} \\ &= 5.4021 \approx 5.4\,\text{units}^2 \quad \text{(1dp)}. \quad \text{A1} \end{aligned}$$

7 **(a)** By Skibidi trigonometry,

$$\begin{aligned} \mathrm{f}(2r-1) - \mathrm{f}(2r+1) &= \cos\left((2r-1)\theta\right) - \cos\left((2r+1)\theta\right) \\ &= -2\sin(2r\theta)\sin(-\theta) \quad \text{M1} \\ &= -2\sin(2r\theta) \cdot -\sin\theta \quad \text{M1} \\ &= 2\sin(2r\theta)\sin\theta. \end{aligned}$$

(b) By part **(a)**,

$$\mathrm{f}(2r-1)-\mathrm{f}(2r+1)=2\sin\theta\sin(2r\theta) \quad\Rightarrow\quad \sin(2r\theta)=\frac{\mathrm{f}(2r-1)-\mathrm{f}(2r+1)}{2\sin\theta}.$$

Taking sigmas on both sides from $r=1$ to $r=n$,

$$\sum_{r=1}^{n}\sin(2r\theta)=\frac{1}{2\sin\theta}\sum_{r=1}^{n}(\mathrm{f}(2r-1)-\mathrm{f}(2r+1)). \quad \text{M1}$$

By the method of differences,

$$\begin{aligned}\sum_{r=1}^{n}(\mathrm{f}(2r-1)-\mathrm{f}(2r+1)) = {} & \mathrm{f}(1)+\mathrm{f}(3)+\cdots+\mathrm{f}(2n-3)+\mathrm{f}(2n-1)\\ & -\mathrm{f}(3)-\mathrm{f}(5)-\cdots-\mathrm{f}(2n-1)-\mathrm{f}(2n+1)\\ = {} & \mathrm{f}(1)-\mathrm{f}(2n+1). \quad \text{M1}\end{aligned}$$

Back-substituting,

$$\begin{aligned}\sum_{r=1}^{n}\sin(2r\theta) &= \frac{1}{2\sin\theta}\left(\mathrm{f}(1)-\mathrm{f}(2n+1)\right)\\ &= \frac{1}{2\sin\theta}\left(\cos\theta-\cos((2n+1)\theta)\right)\\ &= \frac{\cos\theta-\cos((2n+1)\theta)}{2\sin\theta}. \quad \text{M1}\end{aligned}$$

(c) Setting $\theta=\pi/6$, by the double-angle formulae,

$$\begin{aligned}\sum_{r=1}^{n}\sin\left(\frac{\pi r}{6}\right)\cos\left(\frac{\pi r}{6}\right) &= \frac{1}{2}\sum_{r=1}^{n}\sin\left(2r\cdot\frac{\pi}{6}\right) \quad \text{M1}\\ &= \frac{1}{2}\cdot\frac{\cos\frac{\pi}{6}-\cos\left((2n+1)\frac{\pi}{6}\right)}{2\sin\frac{\pi}{6}}\\ &= \frac{1}{2}\cdot\frac{\frac{\sqrt{3}}{2}-\cos\left((2n+1)\frac{\pi}{6}\right)}{2\cdot\frac{1}{2}}\\ &= \frac{\sqrt{3}}{4}-\frac{1}{2}\cos\left((2n+1)\frac{\pi}{6}\right)\\ &= \frac{\sqrt{3}}{4}-\frac{1}{2}\cos\left(\frac{\pi}{6}+\frac{\pi n}{3}\right).\end{aligned}$$

To analyse the cosine term, we observe that

$$c(n):=\cos\left(\frac{\pi}{6}+\frac{\pi n}{3}\right)=\begin{cases}\sqrt{3}/2 & \text{if } n=0,5,\\ 0 & \text{if } n=1,4,\\ -\sqrt{3}/2 & \text{if } n=2,3, \quad \text{M1}\end{cases}$$

and $c(n)=c(n+6)$ for any nonnegative integer n. Thus, the three possible values are given by

$$\begin{aligned}\frac{\sqrt{3}}{4}-\frac{1}{2}\cdot\frac{\sqrt{3}}{2}&=0,\\ \frac{\sqrt{3}}{4}-\frac{1}{2}\cdot 0&=\frac{\sqrt{3}}{4},\\ \frac{\sqrt{3}}{4}-\frac{1}{2}\cdot\left(-\frac{\sqrt{3}}{2}\right)&=\frac{\sqrt{3}}{2}. \quad \text{A1}\end{aligned}$$

8 **(a)** By the chain rule,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= -\sin(1-e^{2x})\cdot(-2e^{2x}) \\ &= -\sin(1-e^{2x})\cdot(-2e^{2x}) \\ &= 2e^{2x}\sin(1-e^{2x}). \quad \text{M1}\end{aligned}$$

By the product rule and algebruh,

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= 2\cdot 2e^{2x}\cdot\sin(1-e^{2x}) + 2e^{2x}\cdot\cos(1-e^{2x})\cdot(-2e^{2x}) \quad \text{M1} \\ &= 4e^{2x}\sin(1-e^{2x}) - 4e^{4x}\cos(1-e^{2x}) \\ &= 2\cdot 2e^{2x}\sin(1-e^{2x}) - 4e^{4x}\cos(1-e^{2x}) \\ &= 2\cdot\frac{\mathrm{d}y}{\mathrm{d}x} - 4e^{4x}y. \\ &= 2\left(\frac{\mathrm{d}y}{\mathrm{d}x} - 2ye^{4x}\right), \quad \text{M1}\end{aligned}$$

where $k = 2$.

(b) Differentiating once more with respect to x,

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 2\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\cdot e^{4x} + y\cdot 4e^{4x}\right)\right). \quad \text{M1}$$

Setting $x = 0$,

$$y = 1, \quad \frac{\mathrm{d}y}{\mathrm{d}x} = 0, \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -4, \quad \frac{\mathrm{d}^3y}{\mathrm{d}x^3} = -24. \quad \text{M1} \quad \text{M1}$$

Therefore, y has a Maclaurin expansion approximation of

$$y \approx 1 + \frac{0}{1!}x + \frac{-4}{2!}x^2 + \frac{-24}{3!}x^3 = 1 - 2x^2 - 4x^3. \quad \text{A1}$$

(c) From part **(b)**, $y \approx 1 - 2x^2$. By the List of Formulae (MF26),

$$\begin{aligned}\frac{1}{\sqrt{a+bx^2}} &= \frac{1}{\sqrt{a}\sqrt{1+\frac{bx^2}{a}}} \\ &= \frac{1}{\sqrt{a}}\left(1+\frac{bx^2}{a}\right)^{-1/2} \\ &\approx \frac{1}{\sqrt{a}}\left(1-\frac{1}{2}\cdot\frac{bx^2}{a}\right) \quad \text{M1} \\ &= \frac{1}{\sqrt{a}} - \frac{b}{2a^{3/2}}x^2.\end{aligned}$$

Comparing coefficients

$$\frac{1}{\sqrt{a}} = 1 \Rightarrow a = 1$$

and

$$-\frac{b}{2a^{3/2}} = -2. \quad \Rightarrow \quad -\frac{b}{2\cdot 1^{3/2}} = -2 \Rightarrow b = 4. \quad \text{A1}$$

9 **(a)** Differentiating with respect to t,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 6t + 2, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = 2t + 6t^2 = t(6t+2). \quad \text{M1}$$

Therefore,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{t(6t+2)}{6t+2} = t, \quad \text{M1}$$

where $k = 1$.

(b) The gradient of the tangent can be equivalently computed using $\tan(60°)$. Hence,

$$t = \frac{\mathrm{d}y}{\mathrm{d}x} = \tan(60°) = \sqrt{3}. \quad \text{M1}$$

Substituting,

$$\begin{aligned} x &= 3(\sqrt{3})^2 + 2(\sqrt{3}) = 3 \cdot 3 + 2\sqrt{3} = 9 + 2\sqrt{3}, \\ y &= (\sqrt{3})^2 + 2(\sqrt{3})^3 = 3 + 2 \cdot 3\sqrt{3} = 3 + 6\sqrt{3}. \quad \text{M1} \end{aligned}$$

Therefore, $P(9 + 2\sqrt{3}, 3 + 6\sqrt{3})$. A1

(c) The required area A is given by

$$A = \int_0^{16} y \, \mathrm{d}x.$$

Since $t \geqslant 0$, using G.C. yields

$$x = 0 \quad \Rightarrow \quad 3t^2 + 2t = 0 \quad \Rightarrow \quad t = 0$$

and

$$x = 16 \quad \Rightarrow \quad 3t^2 + 2t = 16 \quad \Rightarrow \quad t = 2. \quad \text{M1}$$

Therefore,

$$\begin{aligned} A &= \int_0^{16} y \, \mathrm{d}x \\ &= \int_0^2 (t^2 + 2t^3)(6t+2) \, \mathrm{d}t \quad \text{M1} \quad \text{M1} \\ &= 2\int_0^2 t^2(2t+1)(3t+1) \, \mathrm{d}t \\ &= 2\int_0^2 (6t^4 + 5t^3 + t^2) \, \mathrm{d}t \\ &= 2 \cdot \left[6 \cdot \frac{t^5}{5} + 5 \cdot \frac{t^4}{4} + \frac{t^3}{3} \right]_0^2 \quad \text{M1} \\ &= 2 \cdot \left(\frac{916}{15} - 0 \right) = \frac{1832}{15} \text{ units}^2. \quad \text{A1} \end{aligned}$$

10 **(a)** Making the substitution $u = \sqrt{4x+1}$,

$$u = \sqrt{4x+1} \quad \Rightarrow \quad u^2 = 4x+1 \quad \Rightarrow \quad x = \frac{u^2-1}{4} \quad \Rightarrow \quad x-2 = \frac{u^2-9}{4}. \quad \text{M1}$$

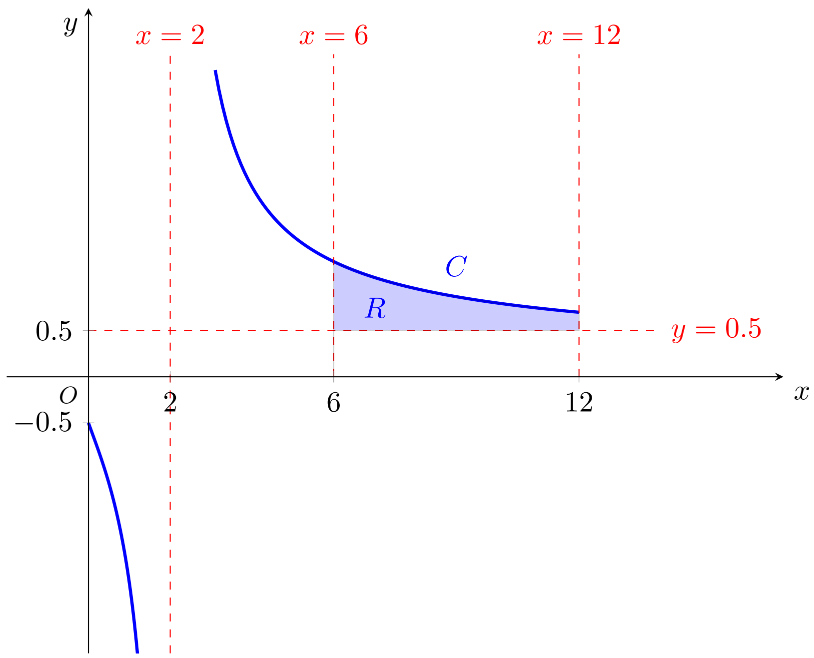
Differentiating,

$$\frac{\mathrm{d}x}{\mathrm{d}u} = \frac{2u}{4} = \frac{u}{2}.$$

Therefore,

$$\begin{aligned}\int \frac{\sqrt{4x+1}}{x-2}\,dx &= \int \frac{u}{\frac{u^2-9}{4}} \cdot \frac{u}{2}\,du \quad \text{M1}\\ &= \int \frac{u}{u^2-9} \cdot \frac{u}{\frac{1}{2}}\,du\\ &= \int \frac{2u^2}{u^2-9}\,du. \quad \text{M1}\end{aligned}$$

(b) **(i)** We sketch as follows:



Smooth curve M1; intercepts and asymptotes M1; region R M1

(ii) By part **(a)**,

$$x = 6 \Rightarrow u = 5 \quad \text{and} \quad x = 12 \Rightarrow u = 7. \quad \text{M1}$$

Hence, the area A of R is given by the integral

$$\begin{aligned}
&\int_6^{12} \frac{\sqrt{4x+1}}{x-2}\,\mathrm{d}x - (12-6)\cdot 0.5 \\
&= \int_5^7 \frac{2u^2}{u^2-9}\,\mathrm{d}u - 3 \quad \text{M1} \\
&= \int_5^7 \left(2 + 18\cdot\frac{1}{u^2-3^2}\right) du - 3 \\
&= \left[2u + \frac{18}{2(3)}\ln\left(\frac{u-3}{u+3}\right)\right]_5^7 - 3 \quad \text{M1} \quad \text{M1} \\
&= \left(14 + 3\ln\left(\frac{4}{10}\right)\right) - \left(10 + 3\ln\left(\frac{2}{8}\right)\right) - 3 \\
&= 1 + 3\ln\left(\frac{4}{10}\cdot\frac{8}{2}\right) \\
&= 4 + 3\ln\left(\frac{8}{5}\right), \quad \text{A1}
\end{aligned}$$

where $a = 1$, $b = 3$, $c = 8/5$.

(iii) Using G.C., the required volume V is given by the integral

$$\begin{aligned}
V &= \pi\int_6^{12}\left(\frac{\sqrt{4x+1}}{x-2}\right)^2 \mathrm{d}x - \pi(0.5)^2\cdot(12-6) \quad \text{M1} \quad \text{M1} \\
&= 11.0432 \approx 11.04\,\text{units}^2. \quad \text{(2dp)} \quad \text{A1}
\end{aligned}$$

11 **(a)** **(i)** Since $v = \dfrac{\mathrm{d}x}{\mathrm{d}t} \Rightarrow \dfrac{\mathrm{d}v}{\mathrm{d}x} = \dfrac{\mathrm{d}^2x}{\mathrm{d}t^2}$,

$$\frac{\mathrm{d}v}{\mathrm{d}t} + \alpha v = 9.8. \quad \text{A1}$$

(ii) Setting $v = 2.4$, $\dfrac{\mathrm{d}v}{\mathrm{d}t} = 5$ yields

$$5 + \alpha\cdot 2.4 = 9.8 \quad \Rightarrow \quad \alpha = 2. \quad \text{A1}$$

(b) By algebruh and integration,

$$\begin{aligned}
\frac{1}{9.8-2v}\frac{\mathrm{d}v}{\mathrm{d}t} &= 1 \\
\int \frac{1}{9.8-2v}\,\mathrm{d}v &= \int 1\,\mathrm{d}t \quad \text{M1} \\
-\frac{1}{2}\ln|9.8-2v| &= t + C \\
\ln|9.8-2v| &= -2t + C \quad \text{M1} \\
9.8 - 2v &= C_1 e^{-2t}, \quad C_1 := \pm e^C \\
v &= 4.9 - \frac{C_1}{2}e^{-2t} \\
v &= 4.9\left(1 - \frac{C_1}{9.8}e^{-2t}\right). \quad \text{M1}
\end{aligned}$$

Since $v = 0$ when $t = 0$, substituting yields $C_1 = 9.8$. M1 Hence,

$$v = 4.9(1 - e^{-2t}), \quad \text{A1}$$

where $A = 4.9$, $B = -2$.

(c) Since $v = \dfrac{\mathrm{d}x}{\mathrm{d}t}$,

$$x = \int v \,\mathrm{d}t = \int 4.9(1 - e^{-2t})\,\mathrm{d}t \quad \text{M1}$$
$$= 4.9\left(t + \frac{1}{2}e^{-2t}\right) + C_2. \quad \text{M1}$$

Since $s = 0$ when $t = 0$, $C_2 = -2.45$. Hence,

$$x = 4.9\left(t + \frac{1}{2}e^{-2t}\right) - 2.45. \quad \text{M1}$$

For the parachutist to reach the ground, $x = 2000$:

$$4.9\left(t + \frac{1}{2}e^{-2t}\right) - 2.45 = 2000. \quad \text{M1}$$

Using G.C., $t = 408.67 \approx 409\,\mathrm{s}$. A1

12 **(a)** **(i)** The aircraft's position has equation

$$L : \mathbf{r} = \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

The air traffic controllers have coordinates $P(0, 0, 0.1)$. At the closest point R,

$$\begin{aligned}
0 &= \overrightarrow{PR} \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \\
&= \left(\begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0.1 \end{pmatrix} \right) \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \quad \text{M1} \\
&= \left(\begin{pmatrix} 4 \\ 3 \\ .9 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right) \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \\
&= \begin{pmatrix} 4 \\ 3 \\ .9 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \quad \text{M1} \\
&= 2.8 + 6\lambda.
\end{aligned}$$

By algebra, $\lambda = -7/15$ so that

$$\overrightarrow{OR} = \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} + \frac{7}{15} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \frac{1}{15} \begin{pmatrix} 53 \\ 52 \\ 1 \end{pmatrix}. \quad \text{M1}$$

Correct to 3 significant figures, $R(3.53, 3.47, 0.0667)$. A1

Alternate solution: Use differentiation to find λ that minimises

$$d := \sqrt{(4 + \lambda)^2 + (3 - \lambda)^2 + (0.9 + 2\lambda)^2}.$$

(ii) We compute the distance

$$PR = \sqrt{(53/15 - 0)^2 + (52/15 - 0)^2 + (1/15 - 0.1)^2} = 4.9500 > 4, \quad \text{M1}$$

thus the air traffic controllers would not be able to see the aircraft. A1

(b) We first divide by 2 to obtain

$$\frac{x-3/2}{3}=\frac{y+1}{2}=\frac{z-k}{-2}.$$

We write the drone's equation in vector form

$$M:\mathbf{r}=\begin{pmatrix}3/2\\-1\\k\end{pmatrix}+\mu\begin{pmatrix}3\\2\\-2\end{pmatrix},\qquad \mu\in\mathbb{R}.$$

When the aircraft and the drone intersect, there exist solutions to the vector equation

$$\begin{pmatrix}4\\3\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\2\end{pmatrix}=\begin{pmatrix}3/2\\-1\\k\end{pmatrix}+\mu\begin{pmatrix}3\\2\\-2\end{pmatrix}.\quad \text{M1}$$

Solving the first two rows yields $\lambda=1.4$, $\mu=1.3$. M1 Substituting into the last row yields

$$1+2\times 1.4=k+1.3\times(-2)\qquad\Longleftrightarrow\qquad k=6.4.$$

Thus, there are no intersections precisely when $k>0$ and $k\neq 6.4$. A1

(c) Let α denote the required acute angle. By the dot product,

$$\begin{aligned}\left|\begin{pmatrix}1\\-1\\2\end{pmatrix}\cdot\begin{pmatrix}3\\2\\-2\end{pmatrix}\right|&=\left|\begin{pmatrix}1\\-1\\2\end{pmatrix}\right|\left|\begin{pmatrix}3\\2\\-2\end{pmatrix}\right|\cos\alpha\\ 3&=\sqrt{6}\sqrt{17}\cos\alpha\\ \cos\alpha&=\frac{3}{\sqrt{6}\sqrt{17}}\quad \text{M1}\\ \alpha&=\cos^{-1}\left(\frac{3}{\sqrt{6}\sqrt{17}}\right)\\ &=72.719^\circ\approx 72.7^\circ.\quad \text{A1}\end{aligned}$$

(d) **(i)** The required distance is

$$d=\sqrt{(4-4.2)^2+(3-0.8)^2+(1-0.2)^2}=2.3494\approx 2.35\,\text{km}.\quad \text{M1}\quad \text{A1}$$

(ii) The vector $\mathbf{v}$ in consideration is

$$\mathbf{v}=\begin{pmatrix}4\\3\\1\end{pmatrix}-\begin{pmatrix}4.2\\0.8\\0.2\end{pmatrix}=\begin{pmatrix}-0.2\\2.2\\0.8\end{pmatrix}.$$

When their distances are minimised $\mathbf{v}\perp\begin{pmatrix}4\\3\\1\end{pmatrix}$. However,

$$\mathbf{v}\cdot\begin{pmatrix}4\\3\\1\end{pmatrix}=\begin{pmatrix}-0.2\\2.2\\0.8\end{pmatrix}\cdot\begin{pmatrix}4\\3\\1\end{pmatrix}=6.6\neq 0.\quad \text{M1}$$

Thus, the computed distance in **(d) (i)** is not the actual shortest distance. A1