H2 MATHEMATICS — 9758/02

Paper 2 — October/November 2024 — 3 hours

Proposed Solutions

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1 **(a)** Since $z = -2 + \mathrm{i}$, $z^* = -2 - \mathrm{i}$. Hence,

$$-1 + \mathrm{i} = a(-2 - \mathrm{i}) + b = (-2a + b) - a\mathrm{i}. \quad \text{M1}$$

Comparing real and imaginary parts, $a = -1$ and $b = -3$. A1

(b) By Skibidi algebruh,

$$\begin{aligned} wz - \frac{w}{z} &= (1 - 3\mathrm{i})(-2 + \mathrm{i}) - \frac{1 - 3\mathrm{i}}{-2 + \mathrm{i}} \\ &= (-2 + \mathrm{i} + 6\mathrm{i} - 3(-1)) - \frac{1 - 3\mathrm{i}}{-2 + \mathrm{i}} \cdot \frac{-2 - \mathrm{i}}{-2 - \mathrm{i}} \quad \text{M1} \\ &= (1 + 7\mathrm{i}) + \frac{(1 - 3\mathrm{i})(2 + \mathrm{i})}{2^2 + 1^2} \quad \text{M1} \\ &= (1 + 7\mathrm{i}) + \frac{2 + \mathrm{i} - 6\mathrm{i} - 3(-1)}{5} \quad \text{M1} \\ &= (1 + 7\mathrm{i}) + \frac{5 - 5\mathrm{i}}{5} = (1 + 7\mathrm{i}) + (1 - \mathrm{i}) = 2 + 6\mathrm{i}, \quad \text{A1} \end{aligned}$$

where $c = 2$ and $d = 6$.

2 The total perimeter is given by

$$3a + 2b = 20 \quad \iff \quad b = 10 - \frac{3}{2}a. \quad \text{M1}$$

The total area R is given by

$$\begin{aligned} R &= ab + \frac{1}{2} \cdot a \cdot a \cdot \sin(\pi/3) \\ &= a \cdot \left(10 - \frac{3}{2}a\right) + \frac{1}{2}a^2 \cdot \frac{\sqrt{3}}{2} \\ &= 10a - \frac{1}{4}(6 - \sqrt{3})a^2. \quad \text{M1} \end{aligned}$$

Differentiating twice,

$$\begin{aligned} \frac{\mathrm{d}R}{\mathrm{d}a} &= 10 - \frac{1}{2}(6 - \sqrt{3})a, \quad \text{M1} \\ \frac{\mathrm{d}^2R}{\mathrm{d}a^2} &= -\frac{1}{2}(6 - \sqrt{3}) < 0, \quad \text{M1} \end{aligned}$$

so that by the second derivative test, R is maximised when

$$\frac{\mathrm{d}R}{\mathrm{d}a} = 0 \quad \iff \quad a = \frac{20}{6 - \sqrt{3}}. \quad \text{M1}$$

Hence, the maximum possible area is

$$R = 10 \cdot \frac{20}{6 - \sqrt{3}} - \frac{1}{4}(6 - \sqrt{3}) \cdot \frac{400}{(6 - \sqrt{3})^2} = \frac{100}{6 - \sqrt{3}} \approx 23.43\,\mathrm{m}^2. \quad \text{(4sf)} \quad \text{A1}$$

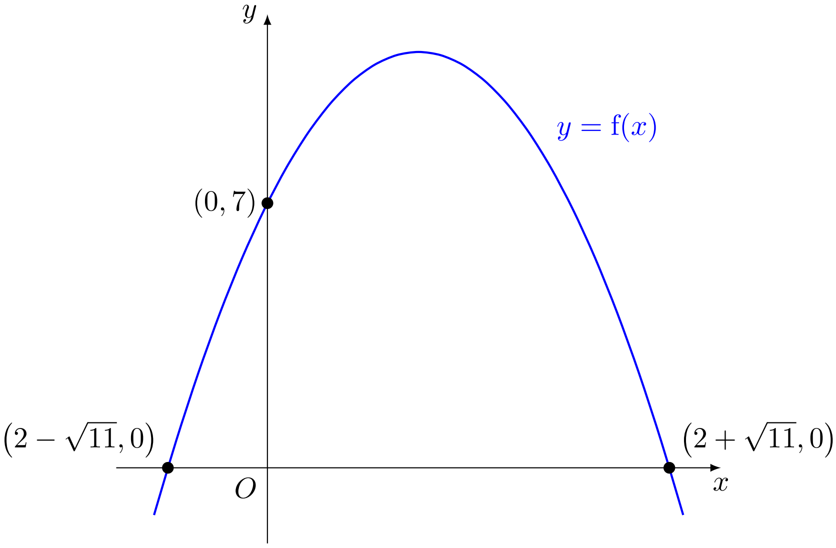
Alternate solution: Accurately finding the maximum point graphically nets full credit.

3 **(a)** Completing the square,

$$\begin{aligned} f(x) &= -x^2 + 4x + 7 \\ &= -x^2 + 4x - 4 + 11 \\ &= -(x-2)^2 + 11. \quad \text{M1} \end{aligned}$$

Thus, $R_f = (-\infty, 11]$. A1

(b) We sketch as follows. Shape M1 ; intercepts M1 .



(c) Since $g' = -2/x^2 < 0$, g is strictly decreasing, so that g is 1-1, and thus g^{-1} exists. B1

(d) Letting $c = g^{-1}(1.5)$ so that $g(c) = 1.5$,

$$\frac{2}{c} + 1 = 1.5 \quad \Longleftrightarrow \quad c = 4. \quad \text{M1}$$

Thus,

$$fg^{-1}(1.5) = f(g^{-1}(1.5)) = f(4) = -(4-2)^2 + 11 = 7. \quad \text{M1} \quad \text{A1}$$

4 **(a)** For $n \geqslant 1$,

$$\begin{aligned} t_n &= (2n^3 - 8n^2 - 4n) - \big(2(n-1)^3 - 8(n-1)^2 - 4(n-1)\big) \\ &= \big(2n^3 - 8n^2 - 4n\big) - \big(2(n^3 - 3n^2 + 3n - 1) - 8(n^2 - 2n + 1) - 4(n-1)\big) \quad \text{M1} \\ &= \big(2n^3 - 8n^2 - 4n\big) - \big(2n^3 - 14n^2 + 18n - 6\big) \\ &= 6n^2 - 22n + 6. \quad \text{M1} \end{aligned}$$

(b) Using G.C.,

$$50n - 204 = 6n^2 - 22n + 6 \quad \text{M1} \quad \Longleftrightarrow \quad n = 5 \text{ or } 7. \quad \text{A1}$$

(c) By baby algebra,

$$50m - 204 > 100 \quad \Longleftrightarrow \quad m \geqslant 7$$

and

$$3n + 16 > 100 \quad \Longleftrightarrow \quad n \geqslant 29.$$

Thus, we want to find integers $m \geqslant 7$ and $n \geqslant 29$ such that

$$50m - 204 = 3n + 16 \quad \Longleftrightarrow \quad 50m = 3n + 220. \quad \text{M1}$$

The smallest possible pair is $m = 8, n = 60$ so that the required number is

$$v_{10} = 3 \cdot 60 + 16 = 196. \quad \text{A1}$$

(d) **(i)** We observe that

$$w_n = 3n^2 - 5n + 7 = n(3n-5) + 7.$$

There are two cases:

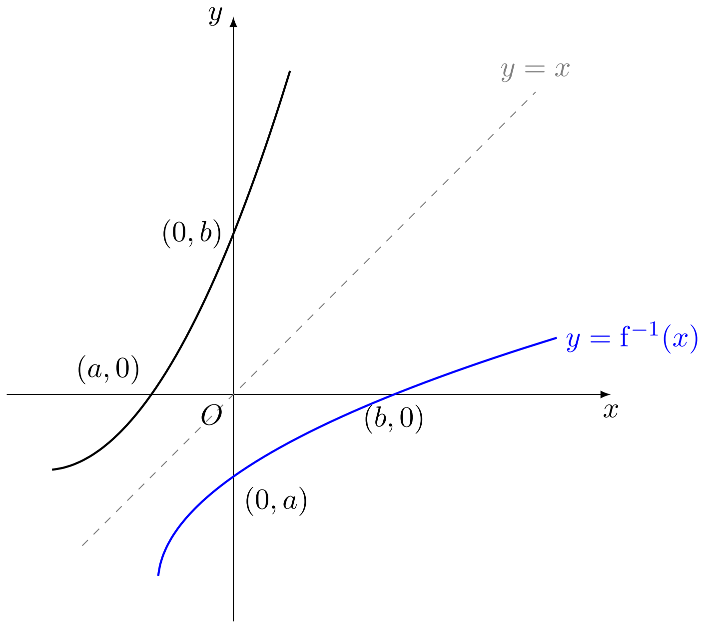
- If n is even, then $n(3n-5)$ is even.
- If n is odd, then $3n-5$ is even and $n(3n-5)$ is even. M1

In any case,

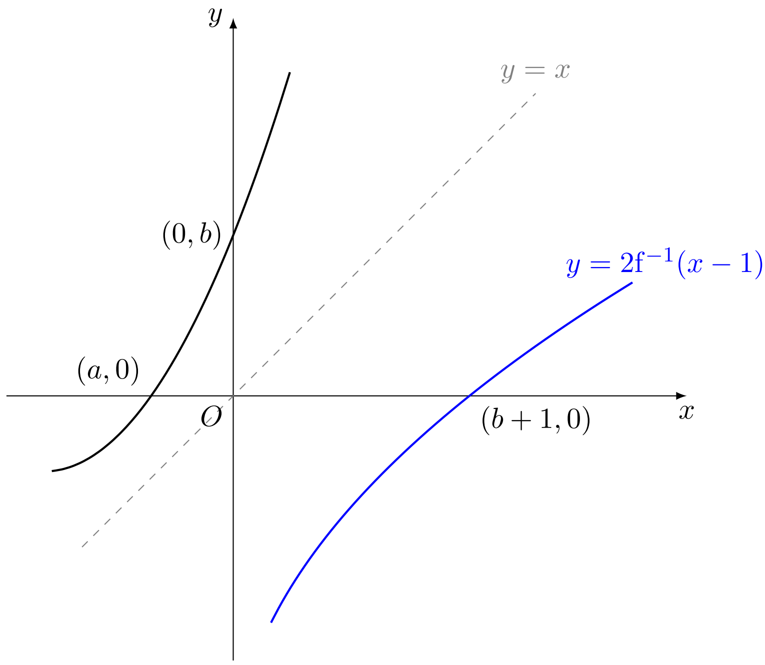
$$w_n = n(3n-5) + 7 = (\text{even}) + (\text{odd}) = (\text{odd}). \quad \text{M1}$$

(ii) For any integer $n \geqslant 1$, $u_n = 50n - 204 = 2(25n - 102)$ is even. If there exists some number x that belongs to U and W, then x is both even and odd, a contradiction. B1

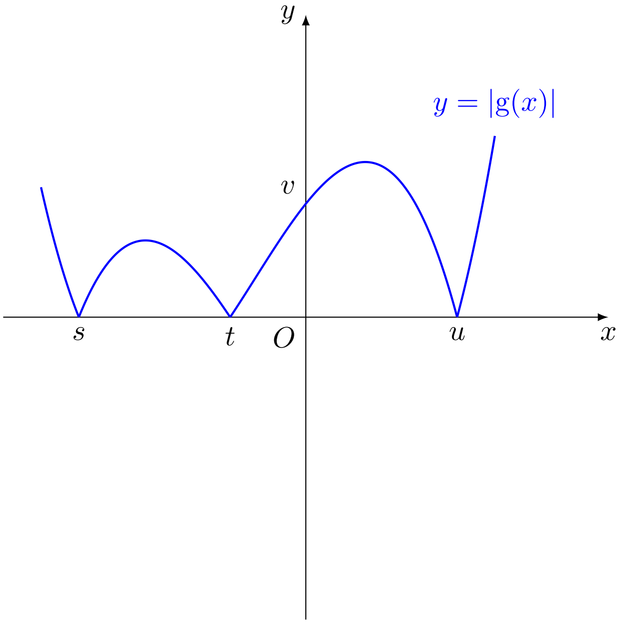
5 **(a)** **(i)** We sketch $y = \mathrm{f}^{-1}(x)$ as a reflection of $y = \mathrm{f}(x)$ about the line $y = x$. M1



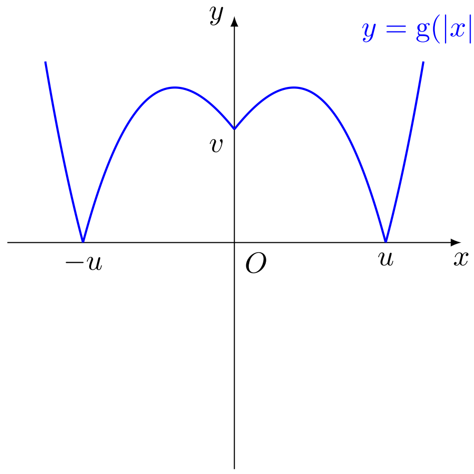
(ii) We sketch $y = 2\mathrm{f}^{-1}(x-1)$. Translation in x M1 ; Scaling in y M1 .



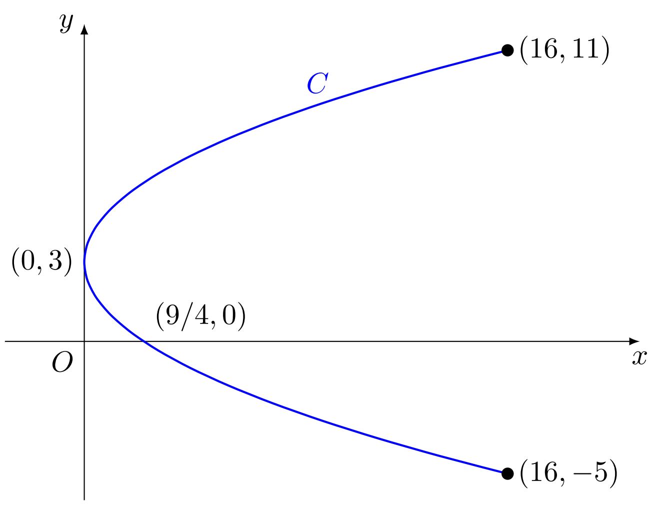
(b) **(i)** We sketch by reflecting the portions where $y \leqslant 0$ about the x-axis. M1 M1



(ii) We sketch by deleting the graph where $x < 0$, and reflecting the portions where $x \geqslant 0$ about the y-axis. M1 M1



(c) **(i)** We sketch as follows. M1



(ii) We indicate the intercepts $(0, 3)$ and $(9/4, 0)$ above. A1 A1

(iii) $y = 3$. A1

6 **(a)** The 65 members are a subset of the total of 90 members, and thus comprise a sample. B1

(b) Choosing a random sample means that each member has an equal probability of being selected. This allows the secretary to minimise biases in opinion from the various members of the club due to differing categories. B1

(c) Let (y, a, s) denote the possible combinations of numbers of individuals that satisfy the constraints

$$y + a + s = 6, \quad y, a, s \geqslant 1, \quad a > y, \quad a > s.$$

The valid combinations of numbers, therefore, are $A \cup B$, where

$$A = \{(2, 3, 1)\}, \quad B = \{(1, 3, 2), (1, 4, 1)\}.$$

The total number T_A of committees corresponding to A is

$$T_A := \binom{27}{2}\binom{45}{3}\binom{18}{1} = 89\,652\,420. \quad \text{M1}$$

The total number T_B of committees corresponding to B is

$$T_B := \binom{27}{1}\binom{45}{3}\binom{18}{2} + \binom{27}{1}\binom{45}{4}\binom{18}{1} = 131\,030\,460. \quad \text{M1}$$

Thus, the required probability is

$$\frac{T_A}{T_A + T_B} = \frac{89\,652\,420}{89\,652\,420 + 131\,030\,460} = \frac{13}{32}. \quad \text{M1} \quad \text{A1}$$

7 **(a)** **(i)** The required probability is

$$\frac{1}{n} + \left(1 - \frac{1}{n}\right) \cdot \frac{1}{n-1} = \frac{2}{n}. \quad \text{M1}$$

(ii) The required probability is

$$\left(1 - \frac{2}{n}\right)\left(\frac{1}{n-2} + \left(1 - \frac{1}{n-2}\right)\frac{1}{n-3}\right) = \frac{n-2}{n} \cdot \left(\frac{1}{n-2} + \frac{n-3}{n-2} \cdot \frac{1}{n-3}\right)$$
$$= \frac{n-2}{n} \cdot \frac{2}{n-2} = \frac{2}{n}. \quad \text{M1}$$

(iii) The probability that either A or B finds the prize is $\frac{2}{n} + \frac{2}{n} = \frac{4}{n}$.
The probability that C finds the prize is

$$1 - \frac{4}{n}. \quad \text{M1}$$

Thus, we want to find n such that

$$1 - \frac{4}{n} > 8 \cdot \frac{4}{n}. \quad \text{M1}$$

Using G.C., $n \in \mathbb{N}, n > 36$. A1

(b) Using the given information,

$$\begin{aligned} \mathbb{P}(Q) &= \mathbb{P}(Q \cap R) + \mathbb{P}(Q \cap R') = x + 0.1 \\ \mathbb{P}(R) &= \mathbb{P}(Q \cap R) + \mathbb{P}(Q' \cap R) = x + \mathbb{P}(Q) = 2x + 0.1, \\ \mathbb{P}(Q \cup R) &= \mathbb{P}(Q) + \mathbb{P}(Q' \cap R) = 2\mathbb{P}(Q) = 2x + 0.2. \quad \text{M1} \end{aligned}$$

Hence,

$$\begin{aligned} \mathbb{P}((Q \cup R)') &= \mathbb{P}(Q) + 2\mathbb{P}(R) \\ 1 - (2x + 0.2) &= (x + 0.1) + 2(2x + 0.1). \quad \text{M1} \end{aligned}$$

Using G.C., $x = 1/14$. A1

8 **(a)** Since Lee suspects that the mean number of birds has decreased (i.e. changed in one direction), Lee should carry out a one-tailed test. B1

(b) Let μ denote the population mean number of birds that visit the feeder during Lee's 20 minutes. B1 The null hypothesis H_0 and alternative hypothesis H_1 are defined by

$$H_0 : \mu = 17.3, \quad H_1 : \mu < 17.3. \quad \text{A1}$$

(c) A direct computation yields $\bar{x} = 512/32 = 16$ and

$$s^2 = \frac{1}{32-1}\left(8702 - \frac{512^2}{32}\right) = 16.451. \quad \text{M1}$$

Assume H_0 holds. Since the number of days on which birds are observed $n = 32 > 30$ is large, by the central limit theorem,

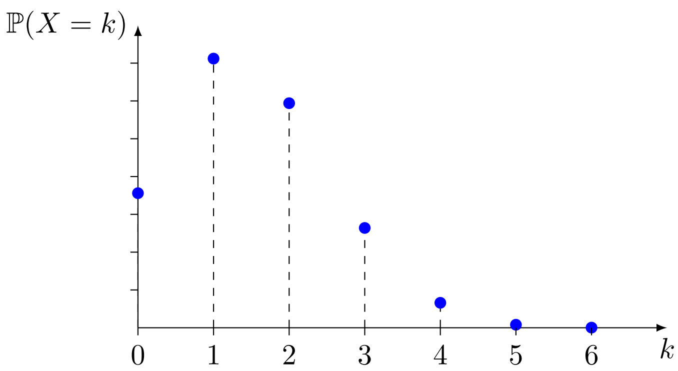
$$\overline{X} \sim \mathcal{N}\left(17.3, \frac{16.451}{32}\right) \quad \text{approximately.}$$

Using G.C., the p-value is computed by $p = \mathbb{P}(\overline{X} < 16) = 0.0349$. M1
To reject H_0, $p < \alpha/100 \Leftrightarrow \alpha > 3.49$. Thus, the minimum integer value for α is $\alpha = 4$. A1

(d) There is sufficient evidence that the average number of birds that visit the feeder during his 20 minutes has decreased. A1

(e) Using only 10 birds would not be sufficient, since we do not have enough samples to invoke the central limit theorem to approximate $\overline{X}$ with a normal distribution. B1

9 **(a)** We sketch as follows. B1 B1



(b) The required estimated probability is

$$\frac{9 \times 1 + 6 \times 2 + 4 \times 3 + 3 \times 4 + 1 \times 5}{4000} = \frac{1}{80}. \quad \text{M1} \quad \text{A1}$$

(c) **(i)** We make the following two assumptions:

- The probability that there is insufficient cheese for each burger is constant. B1
- The amount of cheese in the burgers are independent to one another. B1

(ii) Using G.C., the required probability is $\mathbb{P}(Y < 3) = \mathbb{P}(Y \leqslant 2) = 0.29843 \approx 0.298$. A1

(iii) Let $W \sim \mathrm{B}(28, 0.29843)$ denote the number of days, out of 28, on which fewer than 3 burgers have insufficient cheese in them. M1 Then

$$\mathbb{E}[W] = 28 \cdot 0.29843 = 8.3560 \approx 8.36. \quad \text{A1}$$

(iv) Let $V \sim \mathrm{B}(28 \times 120, 0.03) = \mathrm{B}(3360, 0.03)$ denote the number of burgers, out of $28 \times 120 = 3360$, that have insufficient cheese in them. M1 Then the required probability is

$$\mathbb{P}(V > 100) = 1 - \mathbb{P}(V \leqslant 100) = 0.50577 \approx 0.506. \quad \text{A1}$$

10 Let $X \sim \mathcal{N}(200, 1.6^2)$ denote the mass, in grams, of one plate.

(a) Using G.C., the required probability is $\mathbb{P}(X > 197.5) = 0.94091 \approx 0.941$. A1

(b) Let $Y = 0.95X \sim \mathcal{N}(190, 1.52^2)$ denote the mass of a drilled plate. M1 Using G.C., define the probability p of a drilled plate having masses in grams in $(190, 192)$ by

$$p := \mathbb{P}(190 < Y < 192) = 0.40587. \quad \text{M1}$$

Let $W \sim \mathrm{B}(8, p)$ denote the number of plates, out of 8, whose masses in grams are in $(190, 192)$. M1 The required probability is, using G.C., given by

$$\mathbb{P}(W \geqslant 5) = 1 - \mathbb{P}(W \leqslant 4) = 0.18290 \approx 0.183. \quad \text{A1}$$

(c) Define $V := Y_1 + \cdots + Y_{20} \sim \mathcal{N}(20 \times 190, 20 \times 1.52^2) = \mathcal{N}(3800, 46.208)$. M1 M1
Then the required probability is

$$\mathbb{P}(V < 3805) = 0.76899 \approx 0.769. \quad \text{A1}$$

(d) Defining $S \sim \mathcal{N}(10, 0.3^2)$, $T \sim (44, 1.1^2)$ and $R := V + S_1 + \cdots + S_{80} + T_1 + \cdots + T_{40}$, M1

$$\mathbb{E}[R] = 3800 + 80 \times 10 + 40 \times 44 = 6360,$$
$$\mathrm{Var}(R) = 46.208 + 80 \times 0.3^2 + 40 \times 1.1^2 = 101.808. \quad \text{M1}$$

Let m denote the required mass. By the question,

$$\mathbb{P}(R > m) = 0.05. \quad \text{M1}$$

Using G.C., $m = 6376.596565 \approx 6377$ grams. (nearest gram) A1

11 **(a)** **(i)** The scatter diagram suggests a strong positive curvilinear relationship between m and p. B1

(ii) Let r_1 denote the product moment correlation coefficient between $\ln m$ and p, and r_2 denote the product moment correlation coefficient between $\sqrt{m}$ and p. Using G.C.,

$$r_1 = 0.92851 > 0, \quad r_2 = 0.99143 > 0 \quad \text{M1}$$

Since r_2 is closer to 1, there is stronger correlation between $\sqrt{m}$ and p rather than $\ln m$ and p. Thus, the model $p = d + e\sqrt{m}$ is a better fit. A1 Thus, $r = 0.99143 \approx 0.991$ and $d \approx 32.9$, $e \approx 17.3$ such that, A1 correct to 3 decimal places,

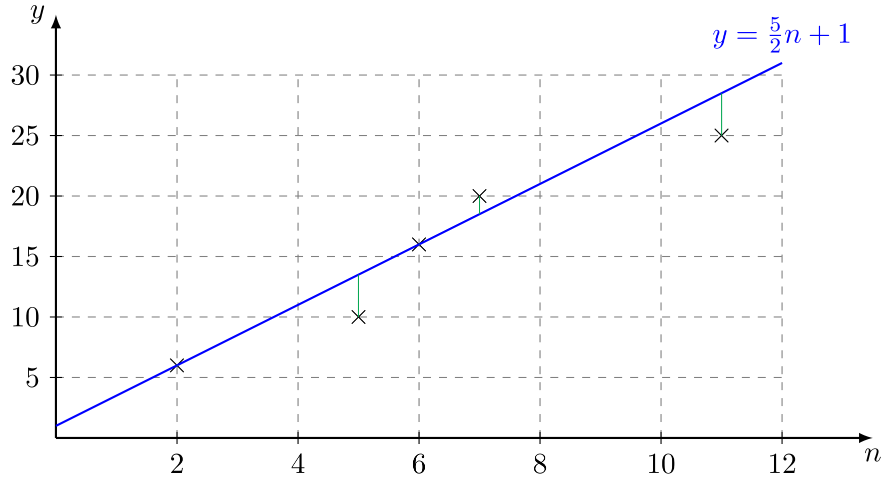
$$p = 32.952 + 17.269\sqrt{m} \approx 33.0 + 17.3\sqrt{m}. \quad \text{A1}$$

(iii) Using G.C.,

$$p = 32.952 + 17.269\sqrt{750} \approx 510. \quad \text{(nearest 10 cents)} \quad \text{A1}$$

Since $750 \in [0, 6000]$ lies inside the data range and $r \approx 0.991$ is close to 1, suggesting a strong positive linear correlation between $\sqrt{m}$ and p, our interpolation estimate of p is reliable. B1

(b) **(i)** We sketch $y = \frac{5}{2}n + 1$ as follows. M1



(ii) **(A)** We mark the residuals above. M1

(B) Since error is modelled to be nonnegative, we square the residuals to obtain computationally convenient nonnegative errors. B1

(C) The sum of the squares of the residuals is

$$0^2 + (-3.5)^2 + 0^2 + 1.5^2 + (-3.5)^2 = 26.75. \quad \text{A1}$$

(iii) A line is a better fit for the data precisely when its total error is smaller, which is equivalent to the sum of the squares of the residuals for Kai's line to be smaller than that of $y = \frac{5}{2}n + 1$. B1

(iv) The sum of the sqaures of the residuals E in terms of c is given by

$$\begin{aligned} E &:= (1-c)^2 + (-2.5-c)^2 + (1-c)^2 + (2.5-c)^2 + (-2.5-c)^2 \\ &= (1 - 2c + c^2) + (6.25 + 5c + c^2) + (1 - 2c + c^2) + (6.25 - 5c + c^2) + (6.25 + 5c + c^2) \\ &= 5c^2 + c + 20.75. \quad \text{M1} \end{aligned}$$

The error for the original line is given by $c = 1$, namely,

$$5(1)^2 + 1 + 20.75 = 26.75.$$

Thus we require c such that

$$5c^2 + c + 20.75 < 26.75 \quad \text{M1} \quad \iff \quad -1.2 < c < 1, \quad c \in \mathbb{R}. \quad \text{A1}$$