

2025 GCE A-Level H2 Mathematics Paper 1

Examination Solutions

Joel Kindiak

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Question 1. Sequences and Series

Solution. (a) Setting $r = 1, 2, 3$,

$$\begin{aligned}u_2 &= 3u_1 + 5 = 3a + 5, \\u_3 &= 3u_2 + 5 = 3(3a + 5) + 5 = 9a + 20, \\u_4 &= 3u_3 + 5 = 3(9a + 20) + 5 = 27a + 65.\end{aligned}$$

(b) Using summation notation,

$$\begin{aligned}110 &= \sum_{r=1}^4 u_r = u_1 + u_2 + u_3 + u_4 \\&= a + (3a + 5) + (9a + 20) + (27a + 65) \\&= 40a + 90.\end{aligned}$$

Solving the equation yields $a = 1/2$.

Question 2. Arithmetic and Geometric Progression

Solution. (a) Let $u_n = a + (n - 1)d$ denote the n -th term of the arithmetic sequence, $n \geq 1$. By considering the sum of the first 20 terms, we have first term a and last term $a + 19d$ so that

$$\frac{20}{2}(a + (a + 19d)) = 65 \quad \Rightarrow \quad 2a + 19d = 6.5.$$

For the next 8 terms, we have first term $a + 20d$ and last term $a + 27d$ so that

$$\frac{8}{2}((a + 20d) + (a + 27d)) = -30 \quad \Rightarrow \quad 2a + 47d = -7.5.$$

Solving the system of linear equations yields $a = 8$ and $d = -1/2$.

(b) By considering the ratio of consecutive terms,

$$\frac{u_{n+1}}{u_n} = \frac{e^{k(n+2)}}{e^{k(n+1)}} = e^{k(n+2)-k(n+1)} = e^k \neq 1,$$

which is constant. Therefore, the sequence having a common ratio e^k is a geometric sequence.

Question 3. Complex Numbers

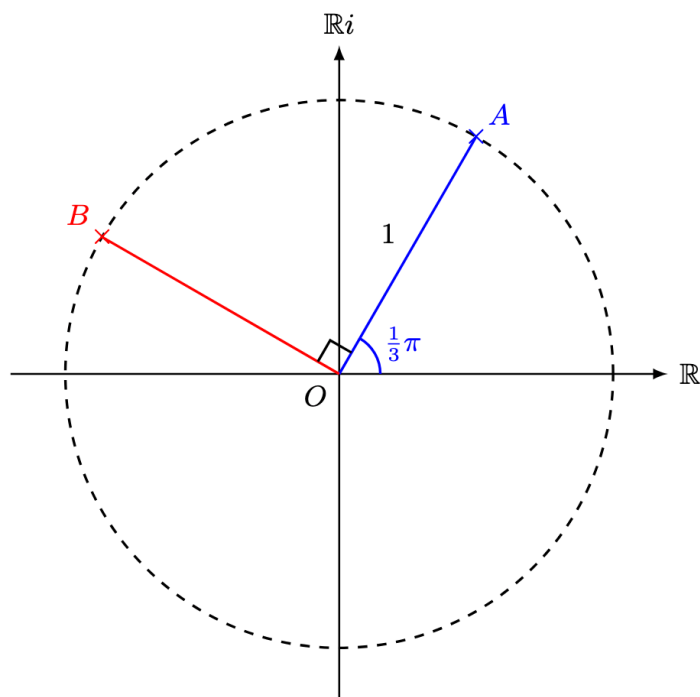
Solution. (a) We first compute the modulus and argument of $\sqrt{3} \pm i$:

$$|\sqrt{3} \pm i| = \sqrt{(\sqrt{3})^2 + (\pm 1)^2} = 2, \quad \arg(\sqrt{3} \pm i) = \pm \frac{1}{6}\pi.$$

Therefore,

$$\begin{aligned} |z| &= \frac{|\sqrt{3} + i|}{|\sqrt{3} - i|} = \frac{2}{2} = 1, \\ \arg(z) &= \arg(\sqrt{3} + i) - \arg(\sqrt{3} - i) \\ &= \frac{1}{6}\pi - \left(-\frac{1}{6}\pi\right) = \frac{1}{3}\pi. \end{aligned}$$

(b) The Argand diagram is drawn below.



The point B represents the complex number iz . Indeed, since $i = e^{i\pi/2}$, we have $|iz| = |z|$ and $\arg(iz) = \arg(z) + \frac{1}{2}\pi$, so multiplying by i rotates A anticlockwise through $\frac{1}{2}\pi$ about O onto B .

Question 4. Optimisation

Solution. (a) We first write the function as $y = (3x + 10)(x + 1)^{-1/2}$. Using the product rule,

$$\begin{aligned} \frac{dy}{dx} &= 3 \cdot (x + 1)^{-1/2} + (3x + 10) \cdot \left(-\frac{1}{2}\right) (x + 1)^{-3/2} \\ &= \frac{1}{2}(x + 1)^{-3/2} (2 \cdot 3 \cdot (x + 1) + (3x + 10) \cdot (-1)) \\ &= \frac{1}{2}(x + 1)^{-3/2} (6x + 6 - (3x + 10)) \\ &= \frac{1}{2}(x + 1)^{-3/2} (3x - 4). \end{aligned}$$

At the turning point, $\frac{dy}{dx} = 0$ so that

$$\frac{1}{2}(x + 1)^{-3/2}(3x - 4) = 0 \quad \Rightarrow \quad x = \frac{4}{3}.$$

(b) We use the product rule again to obtain

$$\frac{d^2y}{dx^2} = \frac{1}{2} \left(-\frac{3}{2} \cdot (x+1)^{-5/2} (3x-4) + (x+1)^{-3/2} \cdot 3 \right).$$

At the turning point, $x = 4/3$ yields

$$\frac{d^2y}{dx^2} \Big|_{x=4/3} = \frac{1}{2} \left(0 + \left(\frac{4}{3} + 1\right)^{-3/2} \cdot 3 \right) = \frac{3}{2} \cdot \left(\frac{7}{3}\right)^{-3/2} \approx 0.421 > 0.$$

Hence, the turning point is a local minimum.

Question 5. Inequalities

Solution. (a) Make the substitution $u = e^x > 0$, so that we solve the inequality

$$\begin{aligned} u &\geq 2u^{-1} + 1 \\ u^2 &\geq 2 + u \\ u^2 - u - 2 &\geq 0 \\ (u - 2)(u + 1) &\geq 0. \end{aligned}$$

Therefore, $u \leq -1$ (which we reject since $u > 0$) or

$$u - 2 \geq 0 \quad \Leftrightarrow \quad u \geq 2 \quad \Leftrightarrow \quad e^x \geq 2 \quad \Leftrightarrow \quad x \geq \ln 2.$$

(b) By part (a),

$$x \geq \ln 2 \quad \Leftrightarrow \quad e^x \geq 2e^{-x} + 1 \quad \Leftrightarrow \quad e^x - 2e^{-x} - 1 \geq 0.$$

We first calculate the indefinite integral for convenience:

$$f(x) := \int (e^x - 2e^{-x} - 1) dx = e^x + 2e^{-x} - x + C.$$

Since $\ln 2 \approx 0.693 \in (0, 1)$, we split the integral into two parts:

$$\begin{aligned} \int_0^1 |e^x - 2e^{-x} - 1| dx &= - \int_0^{\ln 2} (e^x - 2e^{-x} - 1) dx + \int_{\ln 2}^1 (e^x - 2e^{-x} - 1) dx \\ &= -(f(\ln 2) - f(0)) + (f(1) - f(\ln 2)) \\ &= f(1) - 2f(\ln 2) + f(0) \\ &= (e^1 + 2e^{-1} - 1) - 2(e^{\ln 2} + 2e^{-\ln 2} - \ln 2) + (1 + 2 - 0) \\ &= (e + 2/e - 1) - (6 - 2 \ln 2) + 3 \\ &= e + 2/e + 2 \ln 2 - 4. \end{aligned}$$

Question 6. Radioactive Decay

Solution. (a) Using the method of separable variables,

$$\begin{aligned}\int \frac{1}{M} dM &= \int -k dt \\ \ln |M| &= -kt + C \\ M &= \pm e^C \cdot e^{-kt} \\ &= Ae^{-kt}.\end{aligned}$$

By the question, setting $t = 0$ yields

$$M_0 = Ae^{-k \cdot 0} = A \quad \Rightarrow \quad M = M_0 \cdot e^{-kt}.$$

Furthermore, $M = M_0/2$ when $t = 5730$:

$$\begin{aligned}\frac{M_0}{2} &= M_0 \cdot e^{-k \cdot 5730} \\ \frac{1}{2} &= e^{-5730k} \\ k &= \frac{\ln 2}{5730} \approx 1.21 \times 10^{-4}.\end{aligned}$$

(b) Letting T denote the time passed (in years), we solve the equation

$$\begin{aligned}0.2M_0 &= M_0 \cdot e^{-kT} \\ T &= \frac{\ln 0.2}{-k} \\ &= \ln 0.2 \cdot \left(-\frac{5730}{\ln 2}\right) \\ &\approx 13\,304.6 \\ &\approx 13\,300.\end{aligned}$$

Therefore the plant died 13 300 years ago, to the nearest 100 years.

Question 7. Volume of Revolution

Solution. (a) Using linearity and the product rule,

$$\begin{aligned}\frac{d}{dx}(x \ln x - x) &= \frac{d}{dx}(x \ln x) - \frac{d}{dx}(x) \\ &= 1 \cdot \ln x + x \cdot \frac{1}{x} - 1 \\ &= \ln x.\end{aligned}$$

(b) Using (a),

$$\int \ln x \, dx = x \ln x - x + C.$$

At the x -axis, $\ln x = 0 \Rightarrow x = 1$. Therefore, the required volume of revolution V_1 is

$$\begin{aligned}
 V_1 &= \pi \int_1^e (\ln x)^2 dx \\
 &= \pi \left(\left[\underbrace{(x \ln x - x)}_I \cdot \underbrace{\ln x}_S \right]_1^e - \int_1^e \underbrace{(x \ln x - x)}_I \cdot \underbrace{\frac{1}{x}}_D dx \right) \\
 &= \pi \left((0 - 0) - \int_1^e (\ln x - 1) dx \right) \\
 &= \pi \int_1^e (1 - \ln x) dx \\
 &= \pi [x - (x \ln x - x)]_1^e \\
 &= \pi [2x - x \ln x]_1^e \\
 &= \pi \cdot ((2e - e) - (2 - 0)) \\
 &= \pi(e - 2) \text{ units}^3.
 \end{aligned}$$

(c) The curve $y = \ln(x)$ is equivalently given by $x = e^y$. Therefore, the required volume of revolution V_2 is

$$V_2 = \pi e^2 \cdot 1 - \pi \int_0^1 (e^y)^2 dy \approx 13.18 \text{ units}^3 \text{ (2dp)}.$$

Remark 1. The exact volume for V_2 is $V_2 = \frac{1}{2}\pi(e^2 + 1) \text{ units}^3$, left as a simple exercise in integration.

Question 8. Vectors

Solution. (a) We first write π_1 in scalar product form:

$$\pi_1 : \mathbf{r} \cdot \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = 2.$$

The line ℓ perpendicular to π_1 passing through P has vector equation

$$\ell : \mathbf{r} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad \lambda \in \mathbb{R}.$$

The foot of perpendicular F is given by the intersection of ℓ and π_1 :

$$\begin{aligned}
 \left(\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right) \cdot \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} &= 2 \\
 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} &= 2 \\
 -5 + 14\lambda &= 2 \\
 \lambda &= \frac{1}{2}.
 \end{aligned}$$

Therefore,

$$\overrightarrow{OF} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 3/2 \\ 3/2 \end{bmatrix} \Rightarrow F(-2, 3/2, 3/2).$$

(b) Let Q denote the reflection of P in π_1 . By the midpoint theorem,

$$\overrightarrow{OF} = \frac{1}{2}(\overrightarrow{OP} + \overrightarrow{OQ}).$$

Therefore,

$$\overrightarrow{OQ} = 2\overrightarrow{OF} - \overrightarrow{OP} = 2 \begin{bmatrix} -2 \\ 3/2 \\ 3/2 \end{bmatrix} - \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}.$$

(c) The normal vector of π_2 is given by the cross product:

$$\begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} \times \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -13 \\ 4 \\ 6 \end{bmatrix}.$$

The acute angle α between π_1, π_2 is given by the dot product between their normal vectors:

$$\begin{aligned} \left| \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -13 \\ 4 \\ 6 \end{bmatrix} \right| &= \left\| \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\| \left\| \begin{bmatrix} -13 \\ 4 \\ 6 \end{bmatrix} \right\| \cos \alpha \\ 4 &= \sqrt{14} \sqrt{221} \cos \alpha \\ \cos \alpha &= \frac{4}{\sqrt{14} \sqrt{221}} \\ \alpha &\approx 85.9^\circ. \end{aligned}$$

Question 9. Power Series

Solution. (a) Using implicit differentiation,

$$\begin{aligned} \frac{d}{dx}(\sin y) &= \frac{d}{dx}(px) \\ \cos y \cdot \frac{dy}{dx} &= p \\ \frac{dy}{dx} &= \frac{p}{\cos y} \\ &= p \sec y. \end{aligned}$$

(b) Using implicit differentiation two more times,

$$\begin{aligned} \frac{d^2 y}{dx^2} &= p \cdot \sec y \tan y \cdot \frac{dy}{dx} \\ &= \tan y \cdot \left(\frac{dy}{dx} \right)^2 \\ \frac{d^3 y}{dx^3} &= \sec^2 y \cdot \frac{dy}{dx} \cdot \left(\frac{dy}{dx} \right)^2 + \tan y \cdot 2 \cdot \frac{dy}{dx} \cdot \frac{d^2 y}{dx^2} \\ &= \sec^2 y \cdot \left(\frac{dy}{dx} \right)^3 + 2 \tan y \cdot \frac{dy}{dx} \cdot \frac{d^2 y}{dx^2}. \end{aligned}$$

Setting $x = 0$,

$$\begin{aligned}y &= \sin^{-1} 0 = 0, \\ \frac{dy}{dx} &= p \sec 0 = p, \\ \frac{d^2y}{dx^2} &= \tan 0 \cdot p^2 = 0, \\ \frac{d^3y}{dx^3} &= \sec^2 0 \cdot p^3 + 2 \tan 0 \cdot p \cdot 0 = p^3.\end{aligned}$$

Therefore, the Maclaurin series for y up to and including the term in x^3 is given by

$$y \approx 0 + px + \frac{0}{2!} \cdot x^2 + \frac{p^3}{3!} \cdot x^3 = px + \frac{p^3}{6} \cdot x^3.$$

(c) Differentiating term-by-term,

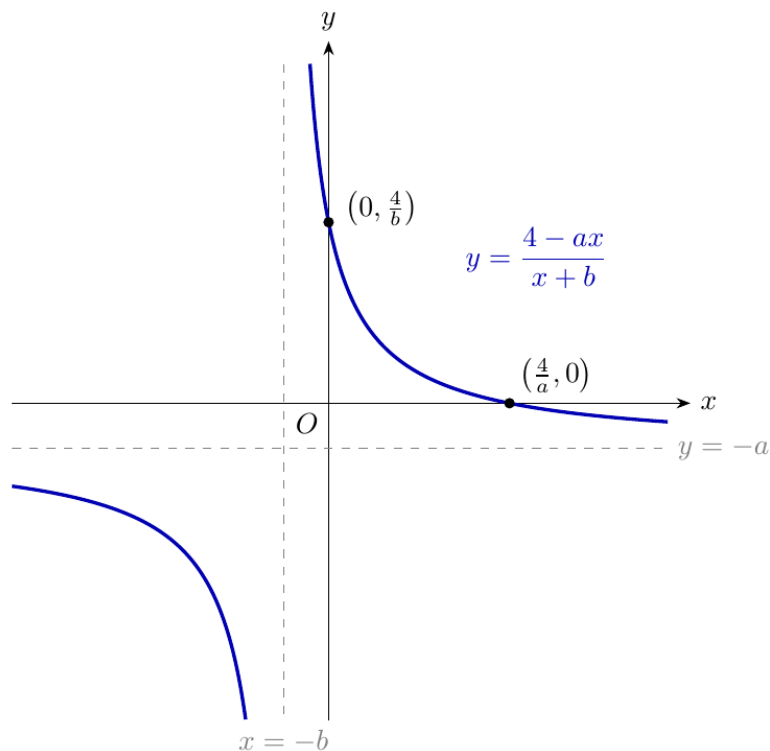
$$\begin{aligned}\frac{d}{dx}(\sin^{-1} px) &\approx \frac{d}{dx} \left(px + \frac{p^3}{6} \cdot x^3 \right) \\ \frac{1}{\sqrt{1 - (px)^2}} \cdot p &\approx p + \frac{p^3}{6} \cdot 3x^2 \\ \frac{1}{\sqrt{1 - p^2x^2}} &\approx 1 + \frac{p^2}{2} \cdot x^2.\end{aligned}$$

Setting $p = 3$ (so that $p^2 = 9$) yields the series expansion

$$\frac{1}{\sqrt{1 - 9x^2}} \approx 1 + \frac{9}{2} \cdot x^2.$$

Question 10. Functions

Solution. (a) The graph has asymptotes $y = -a$, $x = -b$ and axial intercepts $(4/a, 0)$, $(0, 4/b)$.



(b) We first use algebra (or long division) to obtain

$$f(x) = \frac{4 - ax}{x + b} = -a + \frac{4 + ab}{x + b}.$$

Therefore, we make the following transformations:

- Translate by a units in the positive y -direction.
- Scale by a factor of $1/(4 + ab)$ parallel to the y -axis.
- Translate by b units in the positive x -direction.

(c) Denote $y = f^{-1}(x) \Leftrightarrow x = f(y)$ so that

$$\begin{aligned} x &= -a + \frac{4 + ab}{y + b} \\ x + a &= \frac{4 + ab}{y + b} \\ y + b &= \frac{4 + ab}{x + a} \\ y &= -b + \frac{4 + ab}{x + a} \\ f^{-1}(x) &= -b + \frac{4 + ab}{x + a}. \end{aligned}$$

Furthermore, $R_{f^{-1}} = D_f = (-\infty, -b) \cup (-b, \infty) \equiv \mathbb{R} \setminus \{-b\}$.

(d) Given $b = a$,

$$f^{-1}(x) = -a + \frac{4 + a^2}{x + a} = f(x) \quad \Rightarrow \quad f^2(x) = x.$$

Since $2025 = 2 \cdot 1012 + 1$,

$$f^{2025}(2) = f(2) = -a + \frac{4 + a^2}{2 + a} = \frac{4 - 2a}{a + 2}.$$

Question 11. Roller Coaster

Solution. (a) Differentiating with respect to θ ,

$$\begin{aligned} \frac{dx}{d\theta} &= 28 \cdot (\cos \theta + \theta \cdot (-\sin \theta)) \\ &= 28(\cos \theta - \theta \sin \theta), \\ \frac{dy}{d\theta} &= 20 \cdot (-(1 \cdot \sin \theta + \theta \cdot \cos \theta)) \\ &= -20(\sin \theta + \theta \cos \theta). \end{aligned}$$

By an application of the chain rule,

$$\frac{dy}{dx} = \frac{-20(\sin \theta + \theta \cos \theta)}{28(\cos \theta - \theta \sin \theta)} = -\frac{5(\sin \theta + \theta \cos \theta)}{7(\cos \theta - \theta \sin \theta)}.$$

(b) (i) At C and D , the denominator of $\frac{dy}{dx}$ is 0:

$$7(\cos \theta - \theta \sin \theta) = 0 \Rightarrow \cos \theta = \theta \sin \theta.$$

Therefore, $\theta = \pm 0.8603 \approx \pm 0.86$ (2sf). Since $x(-0.8603) \approx 12.3 < 43.7 \approx x(0.8603)$, we have $\theta \approx -0.86$ at C and $\theta \approx 0.86$ at D .

(b) (ii) The x -coordinates of C, D are 12.289 and 43.710 respectively. Hence,

$$CD = 43.710 - 12.289 = 31.421 \approx 31.4 \text{ units.}$$

(c) (i) We solve the equation

$$28(\theta \cos \theta + 1) = 28 \Rightarrow \theta \cos \theta = 0.$$

Therefore, $\theta = 0$ or $\cos \theta = 0 \Rightarrow \theta = \pm \pi/2$. By substituting into expression for y ,

$$y = 40, 10(4 - \pi), 10(4 - \pi)$$

respectively. Hence, $\theta = 0$ at A .

(c) (ii) Furthermore, $\theta = \pm \pi/2$ at B .

(c) (iii) Consequently, $AB = 40 - (10(4 - \pi)) = 10\pi \approx 31.4$ units.

(d) The y -coordinates of E and D are equal:

$$y = 20(2 - 0.8603 \sin 0.8603) \approx 26.956.$$

Therefore, $AE \approx 40 - 26.956 \approx 13.04$. Since

$$\frac{1}{2}CD = \frac{1}{2} \cdot 31.421 = 15.7105,$$

the margin of error ϵ is given by

$$\epsilon := 10\% \cdot \frac{1}{2}CD = 0.1 \cdot 15.7105 = 1.57105.$$

In particular, we require

$$AE \in \left(\frac{1}{2}CD - \epsilon, \frac{1}{2}CD + \epsilon\right) = (14.13945, 17.28155),$$

which does **not** hold. In particular, this section of the track does **not** satisfy the safety condition.

Question 12. Complex Numbers

Solution. (a) Since the coefficients of the given polynomial equation are all real, by the conjugate root theorem, a second root is given by $1 - 2i$.

(b) Given $z = 1 + 2i$, we evaluate z^2 and z^3 :

$$\begin{aligned}z^2 &= 1^2 + 2 \cdot 1 \cdot 2i + (2i)^2 = -3 + 4i, \\z^3 &= 1^3 + 3 \cdot 1^2 \cdot 2i + 3 \cdot 1 \cdot (2i)^2 + (2i)^3 \\&= 1 + 6i - 12 - 8i \\&= -11 - 2i.\end{aligned}$$

Therefore, substituting $z = 1 + 2i$,

$$\begin{aligned}(1 + 2i)^3 + p(1 + 2i)^2 + q(1 + 2i) - 10 &= 0 \\(-11 - 2i) + p(-3 + 4i) + q(1 + 2i) - 10 &= 0 \\(-3p + q - 21) + (-2 + 4p + 2q)i &= 0.\end{aligned}$$

Comparing real and imaginary parts,

$$\begin{aligned}3p - q &= -21, \\4p + 2q &= 2.\end{aligned}$$

Solving the system of linear equations, $p = -4$, $q = 9$.

The given polynomial has a quadratic factor given by

$$\begin{aligned}(z - (1 + 2i))(z - (1 - 2i)) &= z^2 - ((1 + 2i)z + (1 - 2i)z) + (1 + 2i)(1 - 2i) \\&= z^2 - 2z + 5.\end{aligned}$$

Therefore, we factorise

$$z^3 - 4z^2 + 9z - 10 = (z^2 - 2z + 5)(z - 2).$$

Therefore, the third root is $z = 2$.

(c) Since $w = 0$ is not a root (as $1 \neq 0$), we may divide by w^3 on both sides:

$$\left(\frac{1}{w}\right)^3 + p\left(\frac{1}{w}\right)^2 + q \cdot \frac{1}{w} - 10 = 0.$$

Therefore, $1/w = z \Leftrightarrow w = 1/z$. For $z = 1 + 2i$,

$$w = \frac{1}{1 + 2i} \cdot \frac{1 - 2i}{1 - 2i} = \frac{1}{5}(1 - 2i) = \frac{1}{5} - \frac{2}{5}i.$$

For $z = 1 - 2i = (1 + 2i)^*$,

$$w = \frac{1}{1 - 2i} = \left(\frac{1}{1 + 2i}\right)^* = \left(\frac{1}{5} - \frac{2}{5}i\right)^* = \frac{1}{5} + \frac{2}{5}i.$$

Finally, for $z = 2$, $w = 1/2$. Therefore, the three roots are $\frac{1}{5} \pm \frac{2}{5}i$ and $1/2$.

—Joel Kindiak, 4 Nov 25, 1525H