

2023 SACE Stage 2 Mathematical Methods Examination Solutions

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Questions are publicly available. Do comment if you spot any errors.

Remark 1. In SACE Stage 2 Mathematical Methods, Pr is used for probability; we write $\mathbb{P}$ instead.

Question 1. Differentiation Techniques

Solution. (a) By the chain rule,

$$\frac{dy}{dx} = \frac{10x - 2}{5x^2 - 2x}.$$

(b) Writing $y = 4x^{-1} + (7x + 1)^{1/2}$, by the chain rule,

$$\frac{dy}{dx} = -\frac{4}{x^2} + \frac{1}{2}(7x + 1)^{-1/2} \cdot 7 = -\frac{4}{x^2} + \frac{7}{2\sqrt{7x + 1}}.$$

(c) Writing $y = \frac{1}{8}e^{-x} \sin(9x - 3)$, by the product rule and the chain rule,

$$\frac{dy}{dx} = \frac{1}{8}(-e^{-x} \sin(9x - 3) + e^{-x} \cdot 9 \cos(9x - 3)) = \frac{9 \cos(9x - 3) - \sin(9x - 3)}{8e^x}.$$

Question 2. Continuous Random Variables

Solution. (a) (i) Using software (or (c)),

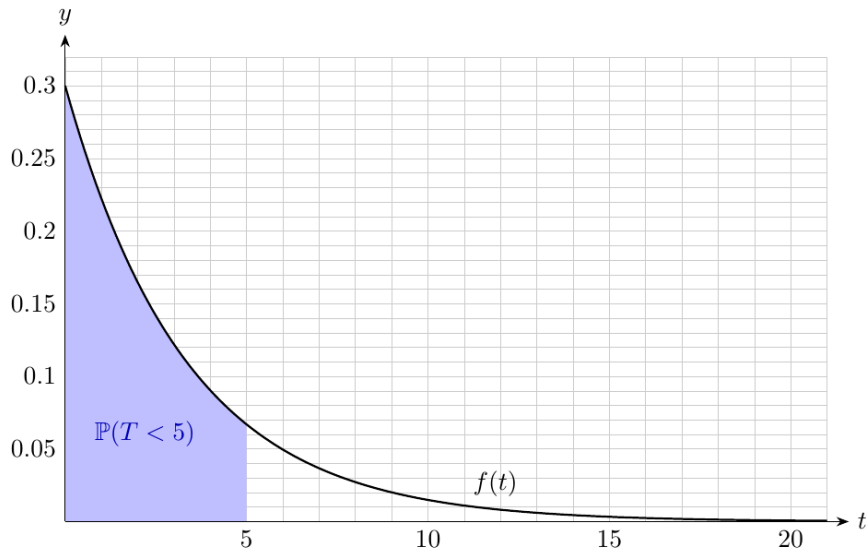
$$\int_2^5 f(t) dt = e^{-0.6} - e^{-1.5} \approx 0.326.$$

(a) (ii) The probability that a randomly chosen customer receives their order between 2 and 5 minutes after placing it is approximately 0.326.

(b) (i) The required probability is

$$\mathbb{P}(T < 5) = \int_0^5 0.3e^{-0.3t} dt = 1 - e^{-1.5} \approx 0.777.$$

(b) (ii) The probability is the area under $y = f(t)$ from $t = 0$ to $t = 5$, shaded below.



(c) By the reverse chain rule,

$$\int 0.3e^{-0.3t} dt = \frac{0.3}{-0.3}e^{-0.3t} + C = -e^{-0.3t} + C.$$

(d) By (c),

$$0.8 = \mathbb{P}(T < r) = \int_0^r 0.3e^{-0.3t} dt = [-e^{-0.3t}]_0^r = 1 - e^{-0.3r}.$$

Therefore,

$$e^{-0.3r} = 0.2 \Rightarrow -0.3r = \ln \frac{1}{5} = -\ln 5 \Rightarrow r = \frac{10}{3} \ln 5.$$

Question 3. Differentiation from First Principles

Solution. (a) For $h \neq 0$,

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{(k(x+h)^2 + 1) - (kx^2 + 1)}{h} \\ &= \frac{k(x^2 + 2xh + h^2) - kx^2}{h} \\ &= \frac{2kxh + kh^2}{h} \\ &= 2kx + kh. \end{aligned}$$

Therefore,

$$f'(x) = \lim_{h \rightarrow 0} (2kx + kh) = 2kx.$$

(b) By (a), $f'(2) = 4k$, and $f(2) = 4k + 1$. Therefore, the tangent at $x = 2$ is

$$y - (4k + 1) = 4k(x - 2) \Rightarrow y = 4kx - 8k + 4k + 1 = 4kx - 4k + 1.$$

(c) Setting $x = 0$, the y -coordinate of A is $-4k + 1$. Therefore, $-4k + 1 = -11$, so that $k = 3$.

Question 4. Binomial Distribution and Confidence Intervals

Solution. (a) (i) Let $X \sim B(10, 0.035)$ denote the number of gold cards in one randomly selected pack.

(a) (i) (1) $\mathbb{E}[X] = 10 \cdot 0.035 = 0.35$.

(a) (i) (2) $\mathbb{P}(X = 1) = 10 \cdot 0.035 \cdot 0.965^9 \approx 0.254$.

(a) (i) (3) Using complements,

$$\mathbb{P}(X > 2) = 1 - \mathbb{P}(X \leq 2) \approx 0.00428.$$

(a) (ii) The 12 packs contain 120 independent cards. Let $Y \sim B(120, 0.035)$ denote the number of gold cards among them. Then

$$\mathbb{P}(Y \geq 1) = 1 - 0.965^{120} \approx 0.986 < 1,$$

so the chance of finding at least one gold card is about 98.6%, not 100%.

(b) (i) The sample contains $1200 \times 10 = 12\,000$ cards, so that

$$\hat{p} = \frac{51}{12\,000} = 0.00425.$$

(b) (ii) Using a 90% confidence level, $z \approx 1.6449$, so that the confidence interval is

$$\begin{aligned} 0.00425 - 1.6449 \sqrt{\frac{0.00425 \cdot 0.99575}{12\,000}} &\leq p \leq 0.00425 + 1.6449 \sqrt{\frac{0.00425 \cdot 0.99575}{12\,000}} \\ 0.00327 &\leq p \leq 0.00523. \end{aligned}$$

(b) (iii) No. The claim is that $p \leq 0.004$, but the confidence interval contains values above 0.004 (indeed, most of the interval lies above 0.004). Hence, we cannot be 90% confident that $p \leq 0.004$, so the interval does not support the claim. (It does not refute the claim either, since it also contains values below 0.004.)

Question 5. Stationary Points and Conjectures

Solution. (a) (i) Writing $f(x) = 2(x-1)^{-1} + 2x$, by the chain rule,

$$f'(x) = -\frac{2}{(x-1)^2} + 2 = \frac{2(x-1)^2 - 2}{(x-1)^2} = \frac{2x^2 - 4x + 2 - 2}{(x-1)^2} = \frac{2x^2 - 4x}{(x-1)^2}.$$

(a) (ii) The stationary points satisfy $f'(x) = 0$. Since $(x-1)^2 > 0$ for $x \neq 1$, we require

$$2x^2 - 4x = 2x(x-2) = 0 \quad \Rightarrow \quad x = 0 \text{ or } x = 2.$$

Both values satisfy $x \neq 1$, so f has exactly two stationary points, at $x = 0$ and $x = 2$.

(b) (i) For $m = 2, 3, 4$, the x -coordinates of the stationary points are $x = 1, 3$, $x = 2, 4$ and $x = 3, 5$ respectively.

(b) (ii) The x -coordinates of the two stationary points of g are $x = m-1$ and $x = m+1$.

(b) (iii) We prove the conjecture. Writing $g(x) = 2(x-m)^{-1} + 2x$, by the chain rule,

$$g'(x) = -\frac{2}{(x-m)^2} + 2.$$

Setting $g'(x) = 0$ and multiplying by $(x-m)^2/2 > 0$,

$$(x-m)^2 = 1 \quad \Rightarrow \quad x-m = \pm 1 \quad \Rightarrow \quad x = m \pm 1.$$

Both values satisfy $x \neq m$. Therefore, for every constant m , g has exactly two stationary points, at $x = m - 1$ and $x = m + 1$.

Remark 2. Equivalently, $g(x) = f(x - m + 1) + 2(m - 1)$, so the graph of g is a translation of the graph of f by $m - 1$ units in the positive x -direction (and $2(m - 1)$ units in the positive y -direction). Translations move stationary points without changing their number.

Question 6. Graphing Techniques

Solution. (a) The function f decreases until B , increases from B to D , and decreases after D . We complete the sign diagram as follows.

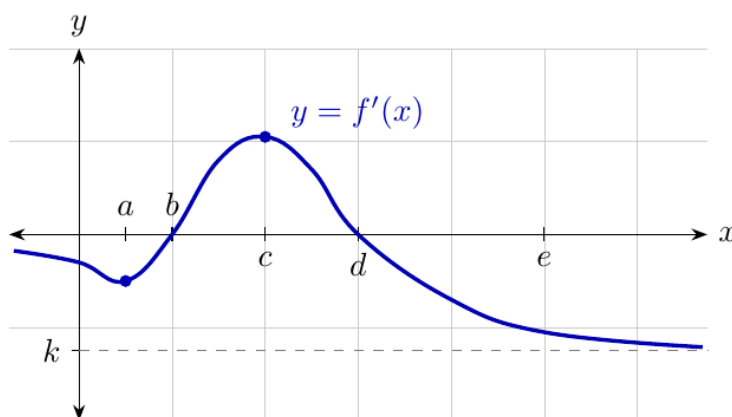
$$f'(x) \quad \leftarrow \begin{array}{ccccccc} & - & & 0 & & + & & 0 & & - \\ & & & b & & & & d & & \end{array} \rightarrow x$$

(b) (i) $f'(0) > f'(a)$. Since $f''(x) = 0$ only at a and c , and f is concave down to the left of a , the gradient f' decreases until $x = a$, where the curve is steepest on its decreasing part.

(b) (ii) $f(d) > 0$ and $f''(d) < 0$, since D is a local maximum above the x -axis.

(c) $f''(x) \geq 0$ for $a \leq x \leq c$, where f is concave up.

(d) The graph of $y = f'(x)$ is negative with a minimum at $x = a$, crosses zero at $x = b$, has a positive maximum at $x = c$, crosses zero at $x = d$, and then decreases towards the horizontal asymptote $y = k$ from above. We sketch as follows.



Question 7. Normal Distribution and Confidence Intervals

Solution. (a) (i) Let $X \sim \mathcal{N}(21.9, 3.26^2)$ denote the daily milk production of a randomly selected cow. Then $\mathbb{P}(X < 23) \approx 0.632$.

(a) (ii) Since $\mathbb{P}(X \geq c) = 0.15$, $\mathbb{P}(X < c) = 0.85$. Using z -scores,

$$\frac{c - 21.9}{3.26} = 1.0364 \quad \Rightarrow \quad c \approx 25.3 \text{ litres.}$$

(b) The daily milk productions of the n randomly chosen cows are independent and each normally distributed. Since a sum of independent normally distributed random variables is normally distributed, S_n is normally distributed (for any n , without needing the central limit theorem).

(c) (i) By the properties of sums of independent random variables,

$$\mu_{S_{90}} = 90 \cdot 21.9 = 1971 \text{ litres}, \quad \sigma_{S_{90}} = \sqrt{90 \cdot 3.26^2} = 3.26\sqrt{90} \approx 30.93 \text{ litres}.$$

(c) (ii) (1) Since $S_{90} \sim \mathcal{N}(1971, 30.93^2)$, $\mathbb{P}(1900 < S_{90} < 1950) \approx 0.238$.

(c) (ii) (2) Similarly, $\mathbb{P}(S_{90} > 2000) \approx 0.174$.

(d) Using a 95% confidence level, $z \approx 1.96$, so that the confidence interval is

$$23.5 - 1.96 \cdot \frac{3.26}{\sqrt{20}} \leq \mu \leq 23.5 + 1.96 \cdot \frac{3.26}{\sqrt{20}} \Rightarrow 22.07 \leq \mu \leq 24.93.$$

(e) Yes. The entire confidence interval lies above the original mean of 21.9 litres, since $22.07 > 21.9$. Hence, with 95% confidence, the new feeding plan has increased the mean daily milk production per cow.

(f) (i) The mean daily milk production per cow would be

$$\frac{2000}{90} \approx 22.2 \text{ litres}.$$

(f) (ii) No. The confidence interval contains values below 22.22 litres (for instance 22.1), since its lower bound 22.07 is less than 22.22. Hence, we cannot be 95% confident that the 90 cows will have a mean total daily milk production of more than 2000 litres.

Question 8. Derivatives of Trigonometric Functions

Solution. (a) (i) $L(0) = 0.75e^0 \cos 0 = 0.75$, so the robot initially leans 0.75 radians to the right.

(a) (ii) By the product rule and the chain rule,

$$L'(t) = 0.75e^{-0.75t} (-0.75 \cos(4t) - 4 \sin(4t)),$$

so that $L'(1.3) \approx 0.900$.

(a) (iii) At 1.3 seconds after release, the angle of lean of the robot is increasing (i.e. the robot is rotating towards the right) at a rate of approximately 0.900 radians per second.

(a) (iv) The robot leans to the left with its maximum angle when $L(t)$ attains its minimum. Using software, this occurs at $t \approx 0.739$ seconds after release (where $L(t) \approx -0.423$).

(b) (i) By the product rule,

$$L'(t) = 0.75 (-e^{-t} \cos t - e^{-t} \sin t) = -0.75e^{-t}(\cos t + \sin t).$$

(b) (ii) Since $e^{-t} > 0$, $L'(t) = 0$ if and only if $\cos t + \sin t = 0$. Since $\cos t = 0$ would force $\sin t = \pm 1 \neq 0$, we may divide by $\cos t$:

$$\tan t = -1 \Rightarrow t = \frac{3\pi}{4} + n\pi, \quad n \in \mathbb{Z}.$$

For $0 \leq t \leq 6$, the exact values are

$$t = \frac{3\pi}{4} \quad \text{and} \quad t = \frac{7\pi}{4},$$

since the next solution $\frac{11\pi}{4} \approx 8.6$ exceeds 6.

Question 9. Discrete Random Variables

Solution. (a) Substituting $x = 0$ and $x = 5$,

$$\mathbb{P}(X = 0) = e^{-2} \approx 0.1353, \quad \mathbb{P}(X = 5) = \frac{e^{-2} \cdot 32}{120} \approx 0.0361.$$

As a check, the seven probabilities in the table sum to 1.

(b) By the chain rule,

$$M'(t) = e^{2e^t-2} \cdot 2e^t = 2e^t e^{2e^t-2}.$$

Therefore,

$$\mu = M'(0) = 2e^0 e^{2e^0-2} = 2 \cdot 1 \cdot e^0 = 2.$$

(c) (i) By the product rule and (b),

$$\begin{aligned} M''(t) &= 2e^t \cdot e^{2e^t-2} + 2e^t \cdot 2e^t e^{2e^t-2} \\ &= 2e^{2e^t-2} (e^t + 2e^{2t}) \\ &= 2e^{2e^t-2} (2e^{2t} + e^t). \end{aligned}$$

(c) (ii) By (c) (i) and (b),

$$M''(0) = 2e^0(2+1) = 6, \quad \sigma = \sqrt{M''(0) - \mu^2} = \sqrt{6 - 2^2} = \sqrt{2}.$$

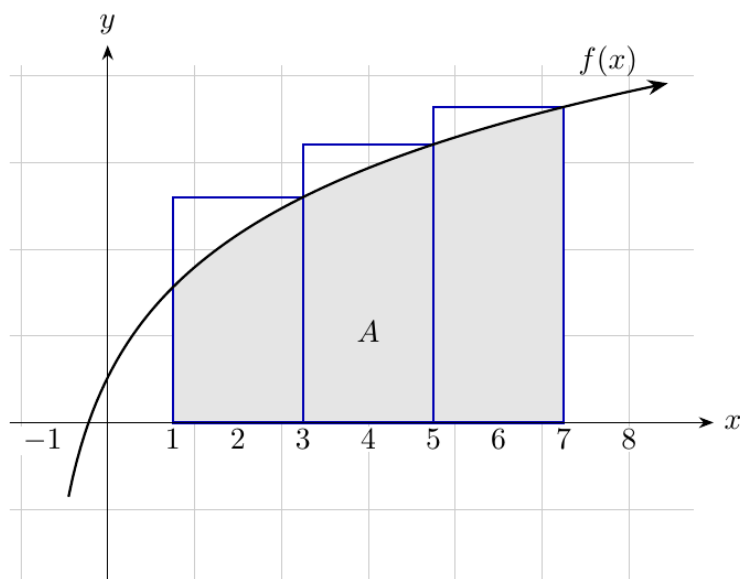
(d) Since $\mu - \sigma = 2 - \sqrt{2} \approx 0.586$ and $\mu + \sigma = 2 + \sqrt{2} \approx 3.41$, the values of X within one standard deviation of the mean are $X = 1, 2, 3$. By (a), the required probability is

$$\mathbb{P}(X = 1) + \mathbb{P}(X = 2) + \mathbb{P}(X = 3) = 0.2707 + 0.2707 + 0.1804 = 0.7218.$$

Remark 3. The random variable X follows a Poisson distribution with parameter 2, whose mean and variance both equal 2. The exact answer to (d) is $\frac{16}{3}e^{-2} \approx 0.722$.

Question 10. Integration Applications

Solution. (a) (i) Since f is increasing, the upper estimate uses the right endpoints of the subintervals $[1, 3]$, $[3, 5]$ and $[5, 7]$. We draw the rectangles as follows.



(a) (ii) Each rectangle has width 2, so that

$$U_3 = 2(f(3) + f(5) + f(7)) = 2(\ln 32 + \ln 72 + \ln 128) \approx 25.2.$$

(a) (iii) The exact value of A is

$$A = \int_1^7 \ln(2(x+1)^2) \, dx.$$

(b) (i) By the product rule,

$$\frac{dy}{dx} = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1.$$

(b) (ii) By (b) (i),

$$\int (\ln x + 1) \, dx = x \ln x + C \quad \Rightarrow \quad \int \ln x \, dx = x \ln x - x + C.$$

For $x > 0$, $\ln(2x^2) = \ln 2 + 2 \ln x$. Therefore,

$$\begin{aligned} \int \ln(2x^2) \, dx &= \int \ln 2 \, dx + 2 \int \ln x \, dx \\ &= x \ln 2 + 2(x \ln x - x) + c \\ &= x(\ln 2 + 2 \ln x - 2) + c \\ &= x(\ln(2x^2) - 2) + c. \end{aligned}$$

(c) Since

$$f(x) = \ln(2(x+1)^2) = g(x+1),$$

the graph of $y = f(x)$ is the graph of $y = g(x)$ translated 1 unit in the negative x -direction.

(d) By (c), the region under f from $x = 1$ to $x = 7$ is the translation of the region under g from $x = 2$ to $x = 8$.
By (b) (ii),

$$\begin{aligned} A &= \int_2^8 \ln(2x^2) \, dx = [x(\ln(2x^2) - 2)]_2^8 \\ &= 8(\ln 128 - 2) - 2(\ln 8 - 2) \\ &= 56 \ln 2 - 16 - 6 \ln 2 + 4 \\ &= 50 \ln 2 - 12. \end{aligned}$$

Therefore, $p = 50$ and $q = -12$.

Remark 4. As a check, $A = 50 \ln 2 - 12 \approx 22.7 < 25.2 \approx U_3$, as expected for an upper estimate.

Remark 5. The diagrams have been drawn and the solutions have been verified with the help of Generative AI.

—Joel Kindiak, 17 Oct 26, 0010H