

2024 SACE Stage 2 Mathematical Methods Examination Solutions

Joel Kindiak

Questions are publicly available. Do comment if you spot any errors.

Remark 1. In SACE Stage 2 Mathematical Methods, $\Pr$ is used for probability; we write $\mathbb{P}$ instead.

Question 1. Normal Distribution

Solution. (a) (i) Since $X \sim \mathcal{N}(54, 6^2)$, the required probability is

$$\mathbb{P}(50 \leq X \leq 52) \approx 0.117.$$

(a) (ii) Similarly,

$$\mathbb{P}(X \geq 58) \approx 0.252.$$

(b) Since $\mathbb{P}(X \geq k) = 0.10$, we have $\mathbb{P}(X < k) = 0.90$. Using z -scores,

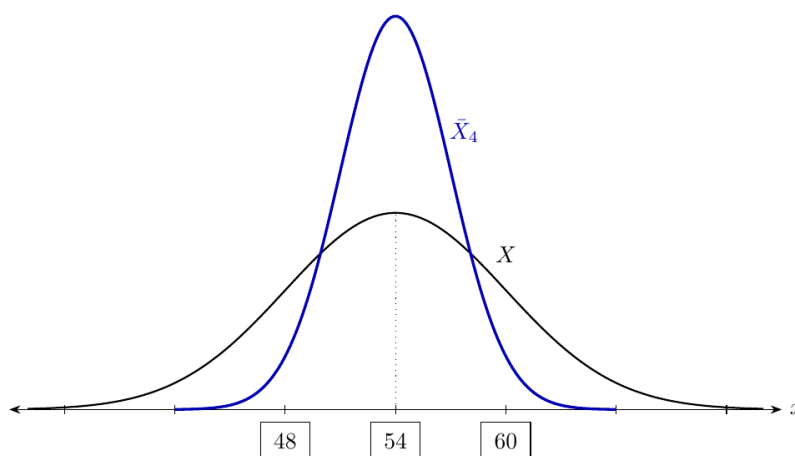
$$\frac{k - 54}{6} = 1.2816 \quad \Rightarrow \quad k = 54 + 6 \cdot 1.2816 = 61.689 \approx 61.7.$$

(c) Since the dashes are one standard deviation apart and the dotted line marks the mean, the three boxes (from left to right) are labelled 48, 54 and 60.

(d) (i) The sample mean of 4 independent observations has

$$\mathbb{E}[\bar{X}_4] = \mu = 54, \quad \sigma_{\bar{X}_4} = \frac{\sigma}{\sqrt{4}} = \frac{6}{2} = 3.$$

(d) (ii) The probability density function of $\bar{X}_4$ has the same centre 54, half the spread, and hence twice the maximum height. We sketch as follows.



Remark 2. Since the area under every probability density function is 1, halving the standard deviation must double the maximum height of the curve.

Question 2. Kinematics

Solution. (a) Substituting $t = 1$,

$$v_A(1) = 3e^{1.5} - 3 \approx 10.4 \text{ m/s.}$$

(b) (i) Integrating with respect to t ,

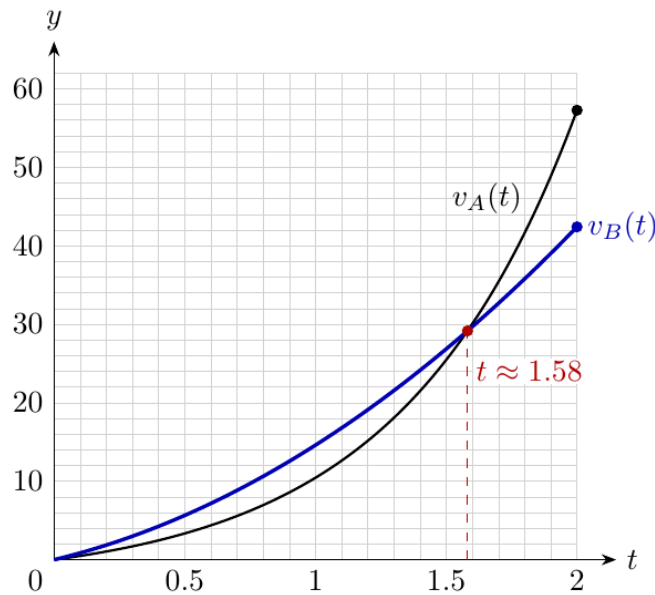
$$s_A(t) = \int (3e^{1.5t} - 3) dt = \frac{3}{1.5}e^{1.5t} - 3t + C = 2e^{1.5t} - 3t + C.$$

Since $s_A(0) = 0$, we have $2 + C = 0$ so that $C = -2$, hence $s_A(t) = 2e^{1.5t} - 3t - 2$, as required.

(b) (ii) By (b) (i),

$$s_A(2) = 2e^3 - 6 - 2 = 2e^3 - 8 \approx 32.2 \text{ m.}$$

(c) We note that $v_B(0) = 0$, $v_B(1) = 14.6$ and $v_B(2) = 42.4$. We sketch as follows.



(d) Solving $v_A(t) = v_B(t)$ using software yields $t = 0$ or $t \approx 1.58$. By (c), the velocity of Car A was greater than that of Car B for $1.58 < t \leq 2$.

(e) Integrating v_B with $s_B(0) = 0$,

$$s_B(2) = \int_0^2 (6.6t^2 + 8t) dt = [2.2t^3 + 4t^2]_0^2 = 17.6 + 16 = 33.6 \text{ m.}$$

Since both cars only travel forwards, the distance each car travels equals its displacement. Since $33.6 > 32.2$, the models suggest that Car B travelled the furthest.

Remark 3. Although Car A ends the 2 seconds much faster, Car B was faster for most of the interval, namely $0 < t < 1.58$. This is why Car B is narrowly ahead.

Question 3. Logarithmic Functions

Solution. (a) By the chain rule, $f'(x) = \frac{1}{x+3}$.

(b) By (a), $f'(0) = \frac{1}{3}$.

(c) Similarly, $g'(x) = \frac{b}{x+3}$, so that $A = (0, \frac{1}{3})$ and $B = (0, \frac{b}{3})$. Since B lies above A ,

$$\frac{b}{3} - \frac{1}{3} = 2 \Rightarrow b - 1 = 6 \Rightarrow b = 7.$$

(d) By (c), $g(x) = 7f(x)$. Therefore, $g(x)$ is obtained from $f(x)$ by a dilation of factor 7 parallel to the y -axis (i.e. from the x -axis).

Question 4. Confidence Intervals

Solution. (a) (i) The sample proportion is $\hat{p} = 23/100 = 0.23$. Using a 95% confidence level, $z \approx 1.96$, so that the confidence interval is

$$0.23 - 1.96\sqrt{\frac{0.23 \cdot 0.77}{100}} \leq p \leq 0.23 + 1.96\sqrt{\frac{0.23 \cdot 0.77}{100}} \Rightarrow 0.148 \leq p \leq 0.312.$$

(a) (ii) Yes.

(a) (iii) The value $p = 0.35$ does not lie in the confidence interval, since $0.35 > 0.312$.

(b) (i) The confidence intervals labelled 18, 40 and 47 do not contain 0.35.

(b) (ii) Only the confidence interval labelled 40 lies entirely above 0.35.

Remark 4. Here $3/50 = 6\%$ of the confidence intervals miss the population proportion, which is close to the 5% that we expect from 95% confidence intervals.

(c) (i) $X \sim B(100, 0.05)$.

(c) (ii) (1) The required probability is

$$\mathbb{P}(X = 5) \approx 0.180.$$

(c) (ii) (2) Using complements,

$$\mathbb{P}(X \geq 1) = 1 - \mathbb{P}(X = 0) = 1 - 0.95^{100} \approx 0.994.$$

(d) Let $Y \sim B(n, 0.05)$ denote the number of n independent 95% confidence intervals that will not contain the population proportion. We require

$$\mathbb{P}(Y \geq 1) = 1 - 0.95^n > 0.5$$

$$0.95^n < 0.5$$

$$n \ln 0.95 < \ln 0.5$$

$$n > \frac{\ln 0.5}{\ln 0.95} \approx 13.5,$$

where the inequality flips since $\ln 0.95 < 0$. Therefore, the minimum number is $n = 14$. As a check, $0.95^{13} \approx 0.513$ while $0.95^{14} \approx 0.488$.

Question 5. Differentiation from First Principles

Solution. (a) For $x \neq 2$ and small $h \neq 0$,

$$\begin{aligned} f(x+h) - f(x) &= \frac{x+h}{x+h-2} - \frac{x}{x-2} \\ &= \frac{(x+h)(x-2) - x(x+h-2)}{(x+h-2)(x-2)} \\ &= \frac{(x^2 - 2x + hx - 2h) - (x^2 + hx - 2x)}{(x+h-2)(x-2)} \\ &= \frac{-2h}{(x+h-2)(x-2)}. \end{aligned}$$

Therefore,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{-2}{(x+h-2)(x-2)} = -\frac{2}{(x-2)^2}.$$

(b) By (a), the slope of the tangent at $x = 6$ is $f'(6) = -\frac{2}{16} = -\frac{1}{8}$.

Question 6. Continuous Random Variables

Solution. (a) Since f is a probability density function,

$$1 = \int_{400}^{440} \frac{k}{x} dx = k [\ln x]_{400}^{440} = k \ln \left(\frac{440}{400} \right) = k \ln 1.1 \quad \Rightarrow \quad k = \frac{1}{\ln 1.1}.$$

(b) (i) Similarly,

$$\int_{400}^{410} \frac{1}{(\ln 1.1)x} dx = \frac{\ln(410/400)}{\ln 1.1} = \frac{\ln 1.025}{\ln 1.1} \approx 0.259.$$

(b) (ii) The probability that a randomly chosen bag contains between 400 and 410 grams of sand (i.e. at most 10 grams of excess sand) is approximately 0.259.

(c) By the definition of the mean,

$$\mu_X = \int_{400}^{440} x \cdot \frac{1}{(\ln 1.1)x} dx = \frac{1}{\ln 1.1} \int_{400}^{440} 1 dx = \frac{440 - 400}{\ln 1.1} = \frac{40}{\ln 1.1} \text{ grams.}$$

(d) (i) Using a 99% confidence level, $z \approx 2.5758$, so that the confidence interval is

$$417 - 2.5758 \cdot \frac{11.5}{\sqrt{25}} \leq \mu \leq 417 + 2.5758 \cdot \frac{11.5}{\sqrt{25}} \quad \Rightarrow \quad 411 \leq \mu \leq 423.$$

(d) (ii) No. By (c), the mean net weight before the alteration is $40/\ln 1.1 \approx 419.7$ grams, which lies inside the confidence interval. Hence, with 99% confidence, the confidence interval does not rule out that the mean net weight is unchanged.

Question 7. Tangents and Conjectures

Solution. (a) Writing $f(x) = x^{-2}$, we have $f'(x) = -2x^{-3}$. Hence, $f(1) = 1$ and $f'(1) = -2$, so that the tangent at $x = 1$ is

$$y - 1 = -2(x - 1) \Rightarrow y = -2x + 3.$$

(b) Setting $y = 0$, $x = \frac{3}{2}$. The x -intercept is $(\frac{3}{2}, 0)$.

(c) The x -intercept of the tangent to $g(x)$ at $x = 1$ is $(\frac{k+3}{2}, 0)$. (This also agrees with (b), where $k = 0$.)

(d) We prove the conjecture. Writing $g(x) = (x^2 + k)^{-1}$, by the chain rule,

$$g'(x) = -(x^2 + k)^{-2} \cdot 2x = -\frac{2x}{(x^2 + k)^2}.$$

Since $k \geq 0$, $1 + k \neq 0$, so that

$$g(1) = \frac{1}{1+k}, \quad g'(1) = -\frac{2}{(1+k)^2}.$$

Therefore, the tangent to $g(x)$ at $x = 1$ has equation

$$y - \frac{1}{1+k} = -\frac{2}{(1+k)^2}(x - 1).$$

Setting $y = 0$ and multiplying both sides by $-(1+k)^2/2$,

$$x - 1 = \frac{1+k}{2} \Rightarrow x = \frac{k+3}{2}.$$

Hence, the x -intercept is $(\frac{k+3}{2}, 0)$ for every non-negative constant k , which proves the conjecture.

Question 8. Integration Applications

Solution. (a) (i) At 9:00 am (9 hours after midnight), vehicles were entering Park-o-Rama at a rate of 50.6 vehicles per hour.

(a) (ii) (1) The graph of $y = L(t)$ passes through P_2 and P_5 , and its maximum occurs at P_5 . Therefore, vehicles were leaving at the maximum rate 17 hours after midnight.

(a) (ii) (2) By Table 2, $\int_0^{24} E(t) dt = 471$ vehicles entered during the 24-hour period.

(a) (ii) (3) By Table 2, the number of vehicles that left during the final 7 hours is

$$\int_{17}^{24} L(t) dt = \int_0^{24} L(t) dt - \int_0^{17} L(t) dt = 406 - 298 = 108.$$

(a) (iii) The number of vehicles in Park-o-Rama at time t is $N(t) = \int_0^t E(s) ds - \int_0^t L(s) ds$, which is maximised when $E(t) = L(t)$ with E dropping below L , i.e. at P_4 . Indeed, $N(15) = 429 - 227 = 202$. Therefore, the maximum occurred 15 hours after midnight.

(b) (i) EVs weigh $2.25 \times 1.36 = 3.06$ tonnes, and the proportion of non-EVs is $1 - 0.07 = 0.93$.

(b) (ii) By (b) (i),

$$\mu_X = 2.25 \cdot 0.93 + 3.06 \cdot 0.07 = 2.3067 \approx 2.31 \text{ tonnes.}$$

(b) (iii) By (a) (iii), there are at most $n = 202$ vehicles in Park-o-Rama at any point during the day. Let T denote the total weight of these 202 vehicles. By the central limit theorem,

$$T \sim \mathcal{N}(202 \cdot 2.31, 202 \cdot 0.207^2) = \mathcal{N}(466.62, 8.6555) \quad \text{approximately.}$$

Hence,

$$\mathbb{P}(T > 475) \approx 0.00220 < 0.005.$$

At any other time, there are fewer vehicles in Park-o-Rama, so the probability that their total weight exceeds 475 tonnes is even smaller. Therefore, the engineers would not be concerned.

Question 9. Exponential Functions

Solution. (a) For P , $f'(x) = e^x - 8e^{2x} = e^x(1 - 8e^x) = 0$ yields $e^x = \frac{1}{8}$, so that

$$y = \frac{1}{8} - 4 \cdot \frac{1}{64} = \frac{1}{16}.$$

For Q , $g'(x) = 3e^{3x} - 8e^{2x} = e^{2x}(3e^x - 8) = 0$ yields $e^x = \frac{8}{3}$, so that

$$y = \left(\frac{8}{3}\right)^3 - 4 \left(\frac{8}{3}\right)^2 = \left(\frac{8}{3}\right)^2 \left(\frac{8}{3} - 4\right) = -\frac{256}{27}.$$

Therefore, the y -coordinates of P and Q are $\frac{1}{16}$ and $-\frac{256}{27}$ respectively.

(b) (i) Differentiating, the stationary points satisfy

$$\begin{aligned} \frac{dy}{dx} &= ae^{ax} - 8e^{2x} = 0 \\ ae^{ax} &= 8e^{2x} \\ e^{(a-2)x} &= \frac{8}{a} \\ (a-2)x &= \ln\left(\frac{8}{a}\right) \\ x &= \frac{1}{a-2} \ln\left(\frac{8}{a}\right), \end{aligned}$$

where we used $8/a > 0$ (since $a > 0$) and $a \neq 2$. Since this equation has exactly one solution, this is the only stationary point.

(b) (ii) (1) Denote the stationary point by x_0 . By (b) (i), $e^{(a-2)x_0} = 8/a$, so that

$$e^{2x_0} = \left(e^{(a-2)x_0}\right)^{\frac{2}{a-2}} = \left(\frac{8}{a}\right)^{\frac{2}{a-2}}, \quad e^{ax_0} = e^{(a-2)x_0} \cdot e^{2x_0} = \frac{8}{a} \left(\frac{8}{a}\right)^{\frac{2}{a-2}}.$$

Therefore,

$$\begin{aligned} y &= e^{ax_0} - 4e^{2x_0} \\ &= \left(\frac{8}{a}\right)^{\frac{2}{a-2}} \left(\frac{8}{a} - 4\right) \\ &= 4 \left(\frac{8}{a}\right)^{\frac{2}{a-2}} \left(\frac{2-a}{a}\right). \end{aligned}$$

(b) (ii) (2) Since $8/a > 0$, $4(8/a)^{\frac{2}{a-2}} > 0$. Hence, the y -coordinate is positive if and only if $\frac{2-a}{a} > 0$. Since $a > 0$, this holds if and only if $2 - a > 0$, i.e. $0 < a < 2$.

Remark 5. As a check, setting $a = 1$ and $a = 3$ in (b) (ii) (1) recovers the y -coordinates $\frac{1}{16} > 0$ and $-\frac{256}{27} < 0$ from (a).

Question 10. Calculus of Exponential and Trigonometric Functions

Solution. (a) At P , $f(x) = f'(x)$:

$$e^{-\cos x} = e^{-\cos x} \sin x \quad \Rightarrow \quad e^{-\cos x}(1 - \sin x) = 0.$$

Since $e^{-\cos x} > 0$, $\sin x = 1$, so that for $0 \leq x \leq 2\pi$, the x -coordinate of P is $x = \frac{\pi}{2}$.

(b) (i) By the product rule and the chain rule,

$$\begin{aligned} f''(x) &= \frac{d}{dx} (e^{-\cos x} \sin x) \\ &= e^{-\cos x} \sin x \cdot \sin x + e^{-\cos x} \cos x \\ &= e^{-\cos x} (\sin^2 x + \cos x) \\ &= e^{-\cos x} (-\cos^2 x + \cos x + 1). \end{aligned}$$

(b) (ii) Setting $f''(x) = f(x)$ and dividing by $e^{-\cos x} > 0$,

$$-\cos^2 x + \cos x + 1 = 1 \quad \Rightarrow \quad \cos x(1 - \cos x) = 0.$$

Therefore, $\cos x = 0$ or $\cos x = 1$, so that $x = 0, \frac{\pi}{2}, \frac{3\pi}{2}, 2\pi$.

(c) By (a), $f'(\frac{\pi}{2}) = f(\frac{\pi}{2})$. By (b) (ii), $f''(\frac{\pi}{2}) = f(\frac{\pi}{2})$. Hence, $f'(\frac{\pi}{2}) = f''(\frac{\pi}{2})$, so that the graphs of $y = f'(x)$ and $y = f''(x)$ intersect at $x = \frac{\pi}{2}$. This is the intersection labelled k in Figure 10, so that $k = \frac{\pi}{2}$.

(d) Let c denote the other intersection point. On region A , $f'(x) \geq f''(x)$ (e.g. $f'(\pi) = 0 > -e = f''(\pi)$), while on region B , $f''(x) \geq f'(x)$ (e.g. $f''(\frac{3\pi}{2}) = 1 > -1 = f'(\frac{3\pi}{2})$). Therefore,

$$\begin{aligned} A - B &= \int_{\pi/2}^c (f'(x) - f''(x)) dx - \int_c^{2\pi} (f''(x) - f'(x)) dx \\ &= \int_{\pi/2}^{2\pi} (f'(x) - f''(x)) dx \\ &= [f(x) - f'(x)]_{\pi/2}^{2\pi} \\ &= (e^{-1} - 0) - (1 - 1) \\ &= \frac{1}{e}. \end{aligned}$$

Therefore, the area of region A exceeds the area of region B by exactly $\frac{1}{e}$ units².

Remark 6. We never needed the value of $c \approx 3.72$: it cancels when the two integrals are combined. Furthermore, the two graphs in Figure 9 touch at P because both their values and their gradients agree there: $f(\frac{\pi}{2}) = f'(\frac{\pi}{2}) = 1$ and, by (c), $f'(\frac{\pi}{2}) = f''(\frac{\pi}{2})$.

Remark 7. The diagrams have been drawn and the solutions have been verified with the help of Generative AI.

—Joel Kindiak, 16 Oct 26, 0138H