

2025 SACE Stage 2 Mathematical Methods Examination Solutions

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Questions are publicly [available](#). Do comment if you spot any errors.

Question 1. Calculus Techniques

Solution. (a) (i) $\frac{dy}{dx} = \frac{1}{x} - \frac{2}{\sqrt{x}}.$

(a) (ii) $\frac{dy}{dx} = -5 \left(\frac{2}{x} + \cos x \right)^4 \left(\frac{2}{x^2} + \sin x \right).$

(a) (iii) $\frac{dy}{dx} = 6x^2 \cdot (3 + e^{6x}) + (2x^3 + 7) \cdot 6e^{6x}.$

(b) $\int \left(\frac{7x+4}{x} \right) dx = \int \left(7 + \frac{4}{x} \right) dx = 7x + 4 \ln |x| + C.$

Question 2. Normal Distribution

Solution. (a) Denoting the weight of an individual small block by $X \sim \mathcal{N}(408, 6.9^2)$, the required probability is

$$\mathbb{P}(X > 400) \approx 0.877.$$

(b) Denoting the weight of an individual large block by $Y \sim \mathcal{N}(624, \sigma^2)$, the required probability is

$$\begin{aligned}\mathbb{P}(Y > 600) &= 0.987 \\ \mathbb{P}\left(\frac{Y - 624}{\sigma} > \frac{600 - 624}{\sigma}\right) &= 0.987 \\ \mathbb{P}\left(Z > -\frac{24}{\sigma}\right) &= 0.987 \\ \mathbb{P}\left(Z \leq -\frac{24}{\sigma}\right) &= 0.013.\end{aligned}$$

Hence, $-24/\sigma = -2.2262$ so that $\sigma = 10.780 \approx 10.8$.

(c) We have $\mathbb{P}(X < 400) \approx 0.123$ and $\mathbb{P}(Y < 600) = 0.013$, so that the small block is more likely to weigh less than its labelled weight.

(d) (i) Denote $\tilde{X} := \sum_{i=1}^{25} X_i$, where $X_1, \dots, X_{25} \sim \mathcal{N}(408, 6.9^2)$ are independent. Then

$$\begin{aligned}\mathbb{E}[\tilde{X}] &= \sum_{i=1}^{25} \mathbb{E}[X_i] = \sum_{i=1}^{25} 408 = 25 \times 408 = 10\,200, \\ \text{Var}(\tilde{X}) &= \sum_{i=1}^{25} \text{Var}(X_i) = \sum_{i=1}^{25} 6.9^2 = 25 \times 6.9^2 = 1190.25.\end{aligned}$$

In particular,

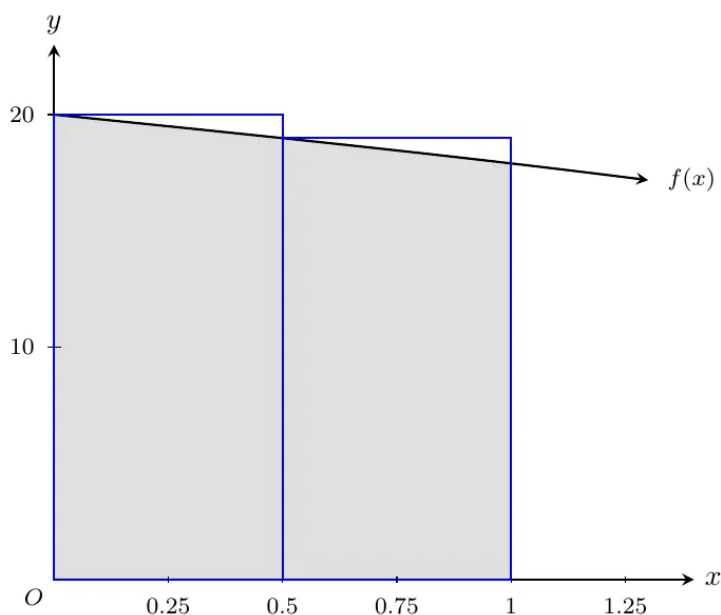
$$\sigma_{\tilde{X}} = \sqrt{\text{Var}(\tilde{X})} = \sqrt{1190.25} = 34.5.$$

(d) (ii) We remark that $\tilde{X} \sim \mathcal{N}(10\,200, 1190.25)$, so that the required probability is

$$\mathbb{P}(\tilde{X} < 10\,000) \approx 3.37 \times 10^{-9}.$$

Question 3. Riemann Sum

Solution. (a) We sketch as follows.



(b) Using part (a),

$$\begin{aligned} U_2 &= f(0) \times 0.5 + f(0.5) \times 0.5 \\ &= 20 \times 0.5 + 18.9875 \times 0.5 \\ &= 19.49375 \approx 19.5. \end{aligned}$$

(c) Differentiating with respect to x twice,

$$f'(x) = -\frac{3}{10}x^2 - 2, \quad f''(x) = -\frac{3}{5}x.$$

(d) (i) $f''(x) \leq 0$ for $0 \leq x \leq 1$.

(d) (ii) By part (d) (i), f is concave down on $[0, 1]$. Therefore,

$$L_2 < \frac{L_2 + U_2}{2} < \int_0^1 f(x) \, dx < U_2.$$

Therefore,

$$\begin{aligned} \left| \int_0^1 f(x) \, dx - U_2 \right| &= U_2 - \int_0^1 f(x) \, dx \\ &< U_2 - \frac{L_2 + U_2}{2} \\ &= \frac{U_2 - L_2}{2} \\ &= \frac{L_2 + U_2}{2} - L_2 \\ &< \int_0^1 f(x) \, dx - L_2 \\ &= \left| \int_0^1 f(x) \, dx - L_2 \right|. \end{aligned}$$

Hence, U_2 is a better approximation for the area of the shaded region than L_2 .

(e) Taking integrals, the required area is

$$\begin{aligned}\int_0^1 \left(-\frac{1}{10}x^3 - 2x + 20 \right) dx &= \left[-\frac{1}{10} \cdot \frac{x^4}{4} - 2 \cdot \frac{x^2}{2} + 20x \right]_0^1 \\ &= \left(-\frac{1}{10} \cdot \frac{1}{4} - 2 \cdot \frac{1}{2} + 20 \cdot 1 \right) - 0 \\ &= 18\frac{39}{40} \text{ units}^2.\end{aligned}$$

Question 4. Binomial Distribution

Solution. (a) (i) Write $X \sim B(5, 1/4)$. Then

$$\mathbb{E}[X] = 5 \cdot \frac{1}{4} = 1.25.$$

(a) (ii) The required probability is

$$\left(1 - \frac{1}{4} \right)^5 = \frac{243}{1024} \approx 0.237.$$

(a) (iii) By hypothesis, $\mathbb{P}(X > k) = 0.1035$. Therefore,

$$\mathbb{P}(X \leq k) = 0.8965.$$

By trial and error, $k = 2$.

(b) (i) The sample proportion is $\hat{p} = 80/1000 = 0.08$.

(b) (ii) We compute

$$\hat{p}(1 - \hat{p}) = 0.08 \cdot 0.92 = 0.0736.$$

Using a confidence interval of 95%, $z \approx 1.96$. Therefore, the confidence interval is given by

$$0.08 - 1.96 \cdot \sqrt{\frac{0.0736}{1000}} \leq p \leq 0.08 + 1.96 \cdot \sqrt{\frac{0.0736}{1000}}.$$

Evaluating to 3 significant figures, $0.0632 \leq p \leq 0.0968$.

(b) (iii) Yes; the upper bound of the confidence interval is less than 0.1035.

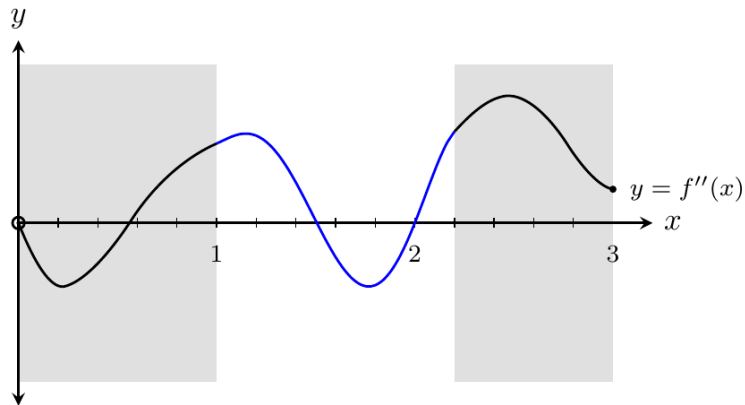
Question 5. Graphing Techniques

Solution. (a) $-$, $+$, $+$.

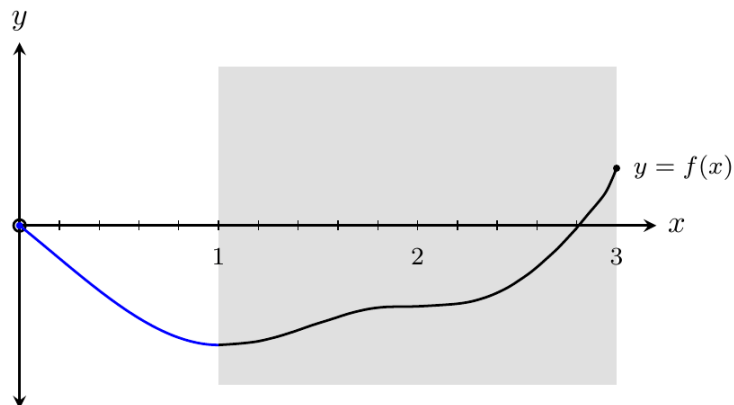
(b) $1 \leq x \leq 3$.

(c) $f(0) > f(1)$.

(d) We sketch as follows, noting that $f''(2) = 0$.



(e) We sketch as follows.



It is possible that $f(0) \neq 0$, but we drew $f(0) = 0$ for simplicity.

Question 6. Differentiation Applications

Solution. (a) By the quotient rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{16x \cdot (x^2 + 20) - 8x^2 \cdot (2x)}{(x^2 + 20)^2} \\ &= \frac{(16x^3 + 320x) - 16x^3}{(x^2 + 20)^2} \\ &= \frac{320x}{(x^2 + 20)^2}.\end{aligned}$$

(b) Denote $w := \frac{dy}{dx}$. Differentiating with respect to x :

$$\begin{aligned}\frac{dw}{dx} &= \frac{320 \cdot (x^2 + 20)^2 - 320x \cdot 2(x^2 + 20) \cdot 2x}{(x^2 + 20)^4} \\ &= \frac{320 \cdot (x^2 + 20) - 1280x^2}{(x^2 + 20)^3} \\ &= \frac{320(20 - 3x^2)}{(x^2 + 20)^3}.\end{aligned}$$

Setting $\frac{dw}{dx} = 0$,

$$20 - 3x^2 = 0 \quad \Rightarrow \quad x^2 = \frac{20}{3} \quad \Rightarrow \quad x = \pm \frac{2\sqrt{15}}{3}.$$

Since $\frac{dw}{dx}$ changes sign from negative to positive at $x = -2\sqrt{15}/3 \approx -2.58$, which lies between H and O , this is where w attains its minimum. Substituting into w ,

$$w\left(-\frac{2\sqrt{15}}{3}\right) = \frac{320 \cdot \left(-\frac{2\sqrt{15}}{3}\right)}{\left(\frac{20}{3} + 20\right)^2} = -\frac{3\sqrt{15}}{10} \approx -1.16 < -1.$$

Therefore, Haz Finder cannot drive from H to O .

(c) The tangent at $x = a$ passes through O if and only if the gradient of OA equals $y'(a)$:

$$\frac{y(a)}{a} = y'(a) \quad \Rightarrow \quad \frac{8a}{a^2 + 20} = \frac{320a}{(a^2 + 20)^2}.$$

Since $a \neq 0$, dividing by $8a$ and multiplying by $(a^2 + 20)^2$,

$$a^2 + 20 = 40 \quad \Rightarrow \quad a^2 = 20.$$

Since $a < 0$, we conclude that $a = -2\sqrt{5}$.

Question 7. Probability Theory

Solution. (a) (i) Using the approximation,

$$\begin{aligned} p := \mathbb{P}(X \leq 2000) &\approx \int_0^{2000} f(x) \, dx \\ &= \int_0^{2000} 0.006e^{12-0.006x} e^{12-0.006x} \, dx \\ &\approx 0.368. \end{aligned}$$

(a) (ii) No, since by (a) (i), $p \approx 0.368 > 0.25$.

(b) Since $\mathbb{E}[X] \approx 2100$, the required probability would be approximately

$$\begin{aligned} \mathbb{P}(X \leq 2100) &\approx \int_0^{2100} f(x) \, dx \\ &= \int_0^{2100} 0.006e^{12-0.006x} e^{12-0.006x} \, dx \\ &\approx 0.578. \end{aligned}$$

(c) (i) 2110 calories.

(c) (ii) The sample mean is given by

$$\bar{x} = 2110 + \frac{168}{2} = 2194 \text{ calories.}$$

Question 8. Discrete Random Variables

Solution. (a) (i) (1) Let X_6 denote the number of points that a player receives. If $X_6 \sim B(n, p)$ for some n and p , then

$$\mathbb{P}(X_6 = n + 1) = 0,$$

a contradiction, since the dice throw outcome

$$(6, 6, \dots, 6, 1)$$

occurs with positive probability.

(a) (i) (2) Every player must throw a die at least once, so that $X_6 \geq 1 > 0$. That is, the set of dice throw outcomes such that $X_6 = 0$ is empty, and thus impossible.

(a) (ii) (1) Setting $x = 2$,

$$\mathbb{P}(X_6 = 2) = \left(\frac{5}{6}\right)^{2-1} \cdot \frac{1}{6} = \frac{5}{36}.$$

(a) (ii) (2) Setting $x = 5, 6$ respectively,

$$\mathbb{P}(X_6 = 5) = \left(\frac{5}{6}\right)^4 \cdot \frac{1}{6}, \quad \mathbb{P}(X_6 = 6) = \left(\frac{5}{6}\right)^5 \cdot \frac{1}{6}.$$

Taking the disjoint union,

$$\begin{aligned}\mathbb{P}(X_6 = 5 \text{ or } 6) &= \left(\frac{5}{6}\right)^4 \cdot \frac{1}{6} + \left(\frac{5}{6}\right)^5 \cdot \frac{1}{6} \\ &= \left(\frac{5}{6}\right)^4 \cdot \frac{1}{6} \cdot \left(1 + \frac{5}{6}\right) \\ &= \frac{6875}{46\,656}.\end{aligned}$$

(b) (i) Taking the disjoint union like before,

$$\begin{aligned}\mathbb{P}(X_n \leq 2) &= \mathbb{P}(X_n = 1) + \mathbb{P}(X_n = 2) \\ &= \frac{1}{n} + \left(1 - \frac{1}{n}\right) \left(\frac{1}{n}\right) \\ &= \left(2 - \frac{1}{n}\right) \cdot \frac{1}{n}.\end{aligned}$$

Setting $n = 10$,

$$\mathbb{P}(X_{10} \leq 2) = \left(2 - \frac{1}{10}\right) \cdot \frac{1}{10} = \frac{19}{100}.$$

(b) (ii) (1) By hypothesis,

$$\begin{aligned}\left(2 - \frac{1}{n}\right) \cdot \frac{1}{n} &= \frac{39}{400} \\ 400 \cdot (2n - 1) &= 39n^2 \\ 800n - 400 &= 39n^2 \\ 39n^2 - 800n + 400 &= 0.\end{aligned}$$

(b) (ii) (2) Solving for n yields $n = 20/39$ or $n = 20$. Since n is a positive integer, we must conclude that $n = 20$.

Question 9. Integration Applications

Solution. (a) 0.

(b) $a = \pi$.

(c) By considering the area under a curve,

$$\begin{aligned}A + B + C &= \int_0^\pi \sin x \, dx \\ 1 + B + 0.566 &= [-\cos x]_0^\pi \\ 1.566 + B &= 2 \\ B &= 0.434 \text{ units}^2.\end{aligned}$$

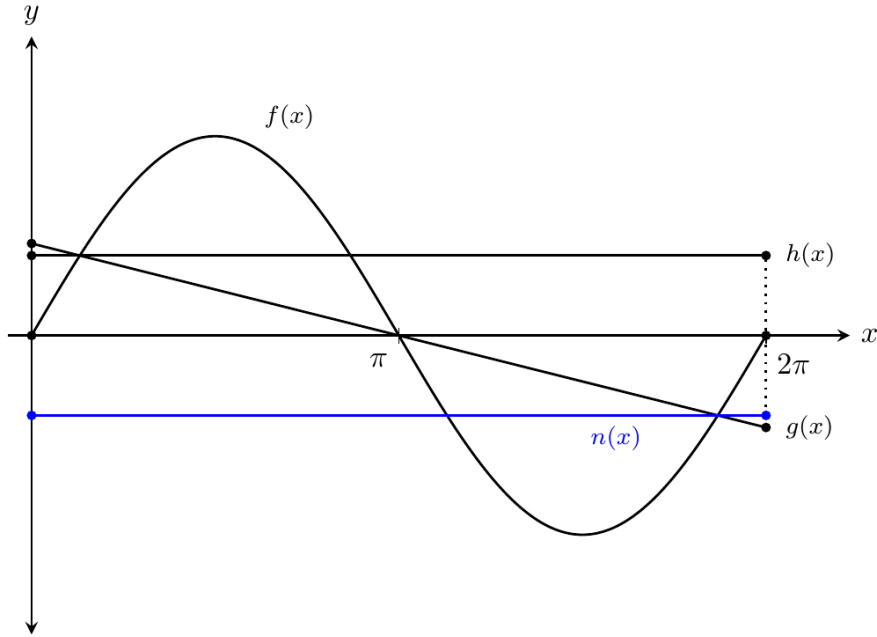
(d) By symmetry, $E + F = 2$. Therefore,

$$\begin{aligned}\int_{p_2}^{2\pi} (h(x) - f(x)) \, dx &= D + E + F \\ &= 1.20 + 2 \\ &= 3.20.\end{aligned}$$

(e) $\int_\pi^{p_3} (g(x) - f(x)) \, dx$.

(f) $\int_0^{2\pi} (f(x) - h(x)) \, dx < 0$.

(g) We sketch as follows, since $n(x) = -h(x)$:



Question 10. Differentiation Applications

Solution. (a) By the chain rule,

$$\begin{aligned} f'(x) &= 100 \cdot \frac{1}{2x - \sin 2x} \cdot (2 - 2 \cos 2x) \\ &= \frac{200 \cdot (1 - \cos 2x)}{2x - \sin 2x}. \end{aligned}$$

(b) By (a), the stationary points are given by $f'(x) = 0$:

$$\frac{200 \cdot (1 - \cos 2x)}{2x - \sin 2x} = 0.$$

For $x > 0$, $\sin 2x < 2x$, so that the denominator is nonzero. Therefore,

$$1 - \cos 2x = 0 \quad \Rightarrow \quad 2x = 2k\pi, \quad k \in \mathbb{Z}^+.$$

In particular, the x -coordinate of A_k is $x_k = k\pi$.

(c) (i) By (b), $\sin 2x_k = \sin 2k\pi = 0$ for each k . Therefore,

$$\begin{aligned} v_k &= f(x_{k+1}) - f(x_k) \\ &= 100 \ln(2x_{k+1} - \sin 2x_{k+1}) - 100 \ln(2x_k - \sin 2x_k) \\ &= 100 \ln \left(\frac{2x_{k+1} - \sin 2x_{k+1}}{2x_k - \sin 2x_k} \right) \\ &= 100 \ln \left(\frac{2(k+1)\pi - 0}{2k\pi - 0} \right) \\ &= 100 \ln \left(\frac{k+1}{k} \right). \end{aligned}$$

(c) (ii) (1) By construction, $h_k = x_{k+1} - x_k = \pi$. By Pythagoras' theorem,

$$\begin{aligned} d_k &= \sqrt{h_k^2 + v_k^2} \\ &= \sqrt{\pi^2 + \left(100 \ln \left(\frac{k+1}{k}\right)\right)^2} \\ &= \sqrt{\pi^2 + 10\,000 \left(\ln \left(\frac{k+1}{k}\right)\right)^2}. \end{aligned}$$

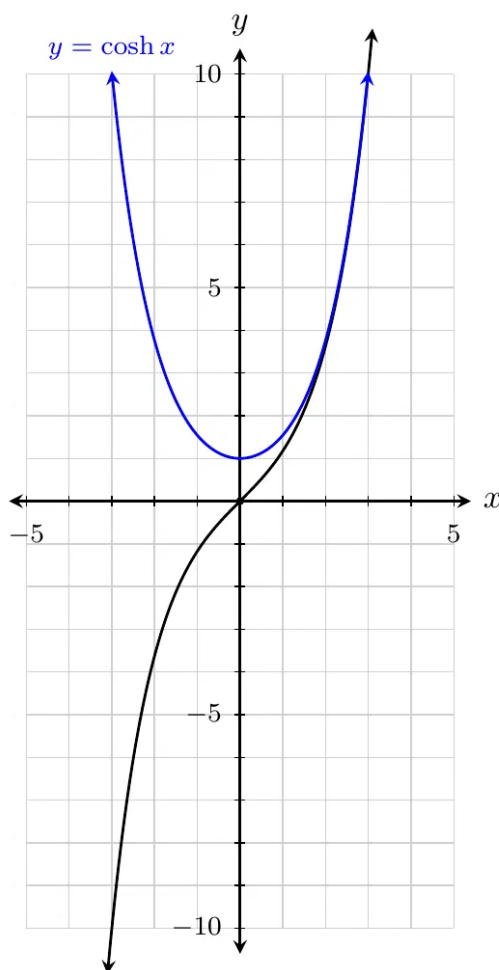
(c) (ii) (2) Since $\ln \left(\frac{k+1}{k}\right)$ decreases as k increases, so does d_k . Setting $d_k < 3.50$,

$$\begin{aligned} \pi^2 + 10\,000 \left(\ln \left(\frac{k+1}{k}\right)\right)^2 &< 3.5^2 \\ \ln \left(1 + \frac{1}{k}\right) &< \sqrt{\frac{3.5^2 - \pi^2}{10\,000}} \approx 0.015429 \\ \frac{1}{k} &< e^{0.015429} - 1 \approx 0.015548 \\ k &> 64.3. \end{aligned}$$

Therefore, the smallest such value is $k = 65$. As a check, $d_{64} \approx 3.503 > 3.50$ while $d_{65} \approx 3.493 < 3.50$.

Question 11. Differentiation Techniques

Solution. (a) We sketch as follows.



(b) Using the quotient rule,

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{d}{dx} \left(\frac{\sinh x}{\cosh x} \right) \\
 &= \frac{\cosh x \cdot \cosh x - \sinh x \cdot \sinh x}{(\cosh x)^2} \\
 &= \frac{(\cosh x)^2 - (\sinh x)^2}{(\cosh x)^2} \\
 &= \frac{(\cosh x + \sinh x)(\cosh x - \sinh x)}{(\cosh x)^2} \\
 &= \frac{e^x \cdot e^{-x}}{(\cosh x)^2} \\
 &= \frac{1}{(\cosh x)^2}.
 \end{aligned}$$

(c) (i) Using the chain rule,

$$\begin{aligned}
 f'(x) &= \frac{13}{(\cosh x)^2} + \frac{10}{\cosh x} \cdot \sinh x \\
 &= 13(\cosh x)^{-2} + 10 \tanh x.
 \end{aligned}$$

Differentiating a second time,

$$\begin{aligned}
 f''(x) &= 13 \cdot (-2)(\cosh x)^{-3} \cdot \sinh x + \frac{10}{(\cosh x)^2} \\
 &= -\frac{26 \sinh x}{(\cosh x)^3} + \frac{10}{(\cosh x)^2} \\
 &= \frac{10 \cosh x - 26 \sinh x}{(\cosh x)^3}.
 \end{aligned}$$

(c) (ii) For this point of inflection, $f''(x) = 0$:

$$\frac{10 \cosh x - 26 \sinh x}{(\cosh x)^3} = 0.$$

Since $\cosh x > 0$ for any real x ,

$$\begin{aligned}
 10 \cosh x - 26 \sinh x &= 0 \\
 \tanh x &= \frac{5}{13} \\
 \frac{e^x - e^{-x}}{e^x + e^{-x}} &= \frac{5}{13} \\
 13e^x - 13e^{-x} &= 5e^x + 5e^{-x} \\
 8e^x &= 18e^{-x} \\
 e^{2x} &= \frac{9}{4} \\
 e^x &= \frac{3}{2} \\
 x &= \ln \left(\frac{3}{2} \right).
 \end{aligned}$$