

2023 SACE Stage 2 Specialist Mathematics Examination Solutions

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Questions are publicly available. Do comment if you spot any errors.

Remark 1. In SACE Stage 2 Specialist Maths, the following equivalent notation is used:

$$\operatorname{cis} \theta \equiv \operatorname{cis}(\theta) \equiv \cos \theta + i \sin \theta.$$

Furthermore, we use row vector notation for vectors:

$$[x, y, z] \equiv \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Question 1. Lines and Planes

Solution. (a) Substituting the parametric equations of l into the equation of P ,

$$(1+t) + 2(2-2t) - (3-t) = 4 \quad \Rightarrow \quad 2-2t = 4 \quad \Rightarrow \quad t = -1.$$

Setting $t = -1$, we obtain $x = 0$, $y = 4$ and $z = 4$. Therefore, P and l intersect at $A(0, 4, 4)$.

(b) The direction vector $[1, -2, -1]$ of l is a normal vector of the required plane. Since the plane passes through A ,

$$x - 2y - z = 0 - 2 \cdot 4 - 4 = -12.$$

Therefore, the required plane is $x - 2y - z = -12$.

Question 2. Related Rates of Change

Solution. (a) $x = 50 \ln(9+1) = 50 \ln 10$ cm.

(b) (i) In the right-angled triangle with vertices B , C and the train, the side opposite θ has length x and the side adjacent to θ has length 50. Since $0 \leq \theta < \frac{\pi}{2}$,

$$\tan \theta = \frac{x}{50} \quad \Rightarrow \quad \theta = \arctan\left(\frac{x}{50}\right).$$

(b) (ii) By the chain rule,

$$\frac{d\theta}{dt} = \frac{d\theta}{dx} \cdot \frac{dx}{dt} = \frac{1/50}{1 + (x/50)^2} \cdot \frac{dx}{dt} = \left(\frac{50}{2500 + x^2} \right) \frac{dx}{dt}.$$

(c) Differentiating $x = 50 \ln(t + 1)$,

$$\frac{dx}{dt} = \frac{50}{t+1} = \frac{50}{9+1} = 5 \text{ cm/s at } t = 9.$$

By (a) and (b) (ii),

$$\frac{d\theta}{dt} = \frac{50}{2500 + (50 \ln 10)^2} \cdot 5 \approx 0.0159 \text{ radians per second.}$$

Question 3. Complex Numbers

Solution. (a) (i) Since z_1 is in the fourth quadrant,

$$|z_1| = \sqrt{3+9} = 2\sqrt{3}, \quad \tan^{-1} \frac{3}{\sqrt{3}} = \frac{\pi}{3} \Rightarrow z_1 = 2\sqrt{3} \operatorname{cis} \left(-\frac{\pi}{3} \right).$$

(a) (ii) Similarly, since z_2 is in the fourth quadrant,

$$|z_2| = \sqrt{27+9} = 6, \quad \tan^{-1} \frac{3}{3\sqrt{3}} = \frac{\pi}{6} \Rightarrow z_2 = 6 \operatorname{cis} \left(-\frac{\pi}{6} \right).$$

(b) (i) Dividing moduli and subtracting arguments,

$$\frac{z_2}{z_1} = \frac{6}{2\sqrt{3}} \operatorname{cis} \left(-\frac{\pi}{6} + \frac{\pi}{3} \right) = \sqrt{3} \operatorname{cis} \frac{\pi}{6}.$$

(b) (ii) By (b) (i), $\theta = \frac{\pi}{6}$.

(c) Since $\frac{3\theta}{2} = \frac{\pi}{4}$, scaling by $\frac{1}{2}$ and rotating anticlockwise through $\frac{\pi}{4}$ amounts to multiplying by $\frac{1}{2} \operatorname{cis} \frac{\pi}{4}$. Hence,

$$z_3 = \frac{1}{2} \operatorname{cis} \frac{\pi}{4} \cdot 6 \operatorname{cis} \left(-\frac{\pi}{6} \right) = 3 \operatorname{cis} \left(\frac{\pi}{4} - \frac{\pi}{6} \right) = 3 \operatorname{cis} \frac{\pi}{12}.$$

Question 4. Mathematical Induction

Solution. (a) For the base case $n = 1$, $4^1 + 15 - 1 = 18 = 9 \cdot 2$, which is divisible by 9. Fix $k \in \mathbb{N}^+$ such that $4^k + 15k - 1 = 9m$ for some integer m , i.e. $4^k = 9m - 15k + 1$. Then

$$\begin{aligned} 4^{k+1} + 15(k+1) - 1 &= 4 \cdot 4^k + 15k + 14 \\ &= 4(9m - 15k + 1) + 15k + 14 \\ &= 36m - 45k + 18 \\ &= 9(4m - 5k + 2), \end{aligned}$$

which is divisible by 9 since $4m - 5k + 2$ is an integer. Since the statement holds for $n = 1$ and it holds for $n = k + 1$ whenever it holds for $n = k$, it holds for all $n \in \mathbb{N}^+$ by mathematical induction.

(b) By (a), $4^n + 15n - 1 = 9m$ for some integer m . Therefore,

$$4^{n+1} + 60n - 1 = 4(4^n + 15n - 1) + 3 = 36m + 3 = 3(12m + 1),$$

which is divisible by 3 since $12m + 1$ is an integer.

Question 5. Systems of Linear Equations

Solution. (a) (i) The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & 4 \\ 1 & -1 & -a & 2 \\ 3 & a & -1 & 8 \end{array} \right].$$

(a) (ii) Applying $R_2 \rightarrow R_1 - R_2$ and $R_3 \rightarrow 3R_1 - R_3$,

$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & 4 \\ 0 & 2 & a-3 & 2 \\ 0 & 3-a & -8 & 4 \end{array} \right].$$

Applying $R_3 \rightarrow 2R_3 - (3-a)R_2$, the third row becomes

$$\begin{aligned} & [0, 2(3-a) - 2(3-a), -16 - (3-a)(a-3) : 8 - 2(3-a)] \\ & = [0, 0, a^2 - 6a - 7 : 2a + 2] \\ & = [0, 0, (a-7)(a+1) : 2(a+1)], \end{aligned}$$

as required.

(b) Figure 4 shows three planes, no two of which are parallel, with no common point of intersection. This corresponds to an inconsistent system, which occurs when $(a-7)(a+1) = 0$ but $2(a+1) \neq 0$, i.e. $a = 7$. (Indeed, for $a = 7$ the normal vectors $[1, 1, -3]$, $[1, -1, -7]$ and $[3, 7, -1]$ are pairwise non-parallel.)

(c) For $a = 3$, the reduced system reads $2y = 2$ and $-16z = 8$, so that

$$y = 1, \quad z = -\frac{1}{2}, \quad x = 4 - 1 - \frac{3}{2} = \frac{3}{2}.$$

Therefore, the three planes intersect at the single point $(\frac{3}{2}, 1, -\frac{1}{2})$.

(d) (i) $a = -1$, since then the third row is entirely zero.

(d) (ii) For $a = -1$, the second row reads $2y - 4z = 2$. Setting $z = t$,

$$y = 1 + 2t, \quad x = 4 - (1 + 2t) + 3t = 3 + t.$$

Therefore, the line is

$$x = 3 + t, \quad y = 1 + 2t, \quad z = t, \quad t \in \mathbb{R},$$

or in vector form, $\mathbf{r} = [3, 1, 0] + t[1, 2, 1]$.

Question 6. Logistic Differential Equations

Solution. (a) (i) Combining the fractions on the right-hand side,

$$\frac{1}{B} + \frac{1}{K-B} = \frac{(K-B) + B}{B(K-B)} = \frac{K}{B(K-B)}.$$

(a) (ii) Rewriting the differential equation as

$$\frac{dB}{dt} = \frac{0.1B(K-B)}{K}$$

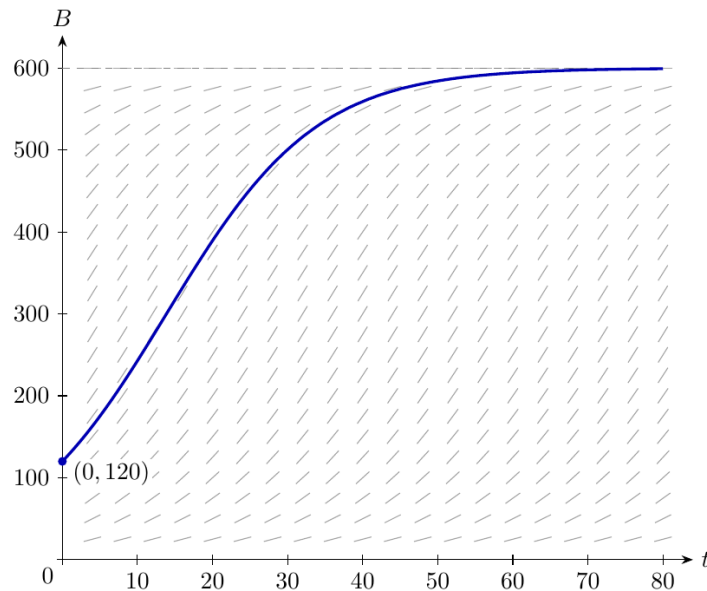
and separating variables, by (a) (i),

$$\begin{aligned}\int \left(\frac{1}{B} + \frac{1}{K-B} \right) dB &= \int 0.1 dt \\ \ln |B| - \ln |K-B| &= 0.1t + C \\ \ln \left| \frac{K-B}{B} \right| &= -0.1t - C \\ \frac{K-B}{B} &= \pm e^{-C} e^{-0.1t} =: A e^{-0.1t}.\end{aligned}$$

Therefore,

$$\frac{K}{B} - 1 = A e^{-0.1t} \quad \Rightarrow \quad B = \frac{K}{1 + A e^{-0.1t}}.$$

(b) The slope field shows horizontal segments at $B = 600$, so $K = 600$ for this butterfly house. The solution curve rises from $(0, 120)$ in an S-shape (steepest at $B = 300$) and approaches $B = 600$ from below. We sketch as follows.



(c) (i) $K = 960$.

(c) (ii) Setting $t = 0$ and $B = 120$,

$$120 = \frac{960}{1 + A} \quad \Rightarrow \quad 1 + A = 8 \quad \Rightarrow \quad A = 7.$$

(c) (iii) Setting $B = 600$,

$$\frac{960}{1 + 7e^{-0.1t}} = 600 \quad \Rightarrow \quad 7e^{-0.1t} = 0.6 \quad \Rightarrow \quad t = 10 \ln \frac{35}{3} \approx 24.6.$$

Therefore, the population exceeds 600 butterflies after approximately 24.6 months.

Question 7. Integration and Volumes of Revolution

Solution. (a) Integrating by parts with $u = x^2$ and $dv = e^x dx$, and using the given integral,

$$\begin{aligned}\int x^2 e^x dx &= x^2 e^x - \int 2x e^x dx \\ &= x^2 e^x - 2(xe^x - e^x) + k \\ &= e^x(x^2 - 2x + 2) + k.\end{aligned}$$

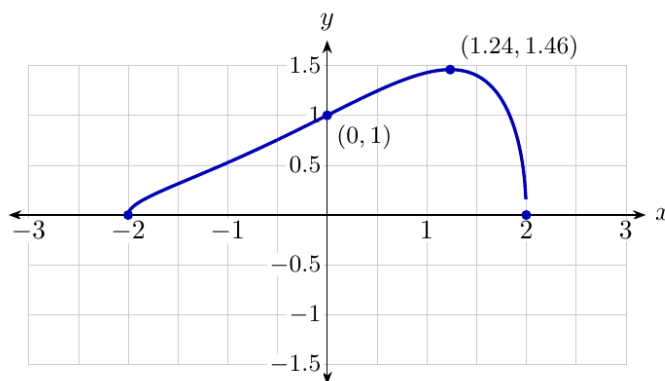
(b) (i) For $-2 \leq x \leq 2$, $g(x) \geq 0$, so that

$$f(g(x)) = \sqrt{\frac{e^x}{4}(4 - x^2)} = \frac{\sqrt{e^x(4 - x^2)}}{\sqrt{4}} = \frac{1}{2}\sqrt{e^x(4 - x^2)}.$$

(b) (ii) The curve meets the x -axis at $x = \pm 2$ and passes through $(0, 1)$. Its maximum occurs where

$$\frac{d}{dx}(e^x(4 - x^2)) = e^x(4 - 2x - x^2) = 0,$$

i.e. at $x = -1 + \sqrt{5} \approx 1.24$, where $y \approx 1.46$. We sketch as follows.



(c) (i) By the volume of revolution formula,

$$V = \pi \int_{-2}^2 (f(g(x)))^2 dx = \pi \int_{-2}^2 \frac{1}{4} e^x (4 - x^2) dx = \frac{\pi}{4} \int_{-2}^2 e^x (4 - x^2) dx.$$

(c) (ii) By (a),

$$\int e^x (4 - x^2) dx = 4e^x - e^x(x^2 - 2x + 2) + k = e^x(2 + 2x - x^2) + k.$$

Therefore, by (c) (i),

$$\begin{aligned}V &= \frac{\pi}{4} [e^x(2 + 2x - x^2)]_{-2}^2 \\ &= \frac{\pi}{4} (e^2(2 + 4 - 4) - e^{-2}(2 - 4 - 4)) \\ &= \frac{\pi}{4} (2e^2 + 6e^{-2}) \\ &= \frac{\pi}{2} (e^2 + 3e^{-2}).\end{aligned}$$

Question 8. Vector Geometry

Solution. (a) In the right-angled triangle ABT , the side AT is opposite θ and BA is the hypotenuse, so that $|\vec{AT}| = |\vec{BA}| \sin \theta$. Since $\mathbf{l}$ points along $\vec{BT}$, the angle between $\vec{BA}$ and $\mathbf{l}$ is θ . Therefore,

$$|\vec{AT}| = |\vec{BA}| \sin \theta = \frac{|\vec{BA}| |\mathbf{l}| \sin \theta}{|\mathbf{l}|} = \frac{|\vec{BA} \times \mathbf{l}|}{|\mathbf{l}|}.$$

(b) $\vec{BA} = [4, 0, 2] - [-1, -6, -2] = [5, 6, 4].$

(c) (i) Computing the cross product,

$$\vec{BA} \times \mathbf{l} = [5, 6, 4] \times [2, 2, -1] = [-6 - 8, 8 + 5, 10 - 12] = [-14, 13, -2],$$

so that $|\vec{BA} \times \mathbf{l}| = \sqrt{196 + 169 + 4} = \sqrt{369} = 3\sqrt{41}.$

(c) (ii) Since $|\mathbf{l}| = \sqrt{4 + 4 + 1} = 3$, by (a),

$$|\vec{AT}| = \frac{3\sqrt{41}}{3} = \sqrt{41}.$$

(c) (iii) $\mathbf{r} = [-1, -6, -2] + \lambda[2, 2, -1], \lambda \in \mathbb{R}.$

(c) (iv) Substituting $[x, y, z] = [-1 + 2\lambda, -6 + 2\lambda, -2 - \lambda]$ into P_1 ,

$$2(-1 + 2\lambda) + 2(-6 + 2\lambda) - (-2 - \lambda) = 6 \Rightarrow 9\lambda - 12 = 6 \Rightarrow \lambda = 2.$$

Setting $\lambda = 2$, the intersection point is $T(3, -2, -4).$

(d) Since $\vec{BT}$ and $\vec{CV}$ are both normal to P_1 , they are parallel, and T, V both lie on P_1 . Hence, the triangles ATB and AVC are similar, so that

$$\frac{|AC|}{|AB|} = \frac{|AV|}{|AT|} = 5 \Rightarrow \vec{AC} = 5\vec{AB},$$

since B and C lie on the same side of A . Therefore,

$$\vec{OC} = [4, 0, 2] + 5[-5, -6, -4] = [-21, -30, -18].$$

Since C lies on P_3 , $k = 2(-21) + 2(-30) - (-18) = -84.$

Remark 2. As a check, the distances from P_1 to P_2 and to P_3 are

$$\frac{6 - (-12)}{3} = 6 \quad \text{and} \quad \frac{6 - (-84)}{3} = 30 = 5 \cdot 6$$

respectively, as expected from the similar triangles.

Question 9. Complex Roots of Unity

Solution. (a) Let $z = r \operatorname{cis} \alpha$ with $r > 0$ and $-\pi < \alpha \leq \pi$. By De Moivre's theorem,

$$r^5 \operatorname{cis} 5\alpha = 1 = \operatorname{cis}(2k\pi),$$

so that $r = 1$ and $\alpha = \frac{2k\pi}{5}$ for $k = 0, \pm 1, \pm 2$. Matching the positions in Figure 11,

$$z_1 = 1, \quad z_2 = \text{cis } \frac{2\pi}{5}, \quad z_3 = \text{cis } \frac{4\pi}{5}, \quad z_4 = \text{cis } \left(-\frac{4\pi}{5}\right), \quad z_5 = \text{cis } \left(-\frac{2\pi}{5}\right).$$

(b) (i) Since $|z| = 1$, $|z^n| = |z|^n = 1$ for every integer n . By the triangle inequality,

$$|f(z)| \leq |z^2| + |z| + 1 + \left|\frac{1}{z}\right| + \left|\frac{1}{z^2}\right| = 1 + 1 + 1 + 1 + 1 = 5.$$

(b) (ii) $z = 1$, since $f(1) = 5$.

(c) (i) Expanding,

$$z^2(z-1)f(z) = (z-1)(z^4 + z^3 + z^2 + z + 1) = z^5 - 1,$$

where the final equality follows from the geometric series (or by expanding and cancelling).

(c) (ii) By (c) (i), the zeros of f are the zeros of $z^5 - 1$ other than $z = 1$ (and $z \neq 0$). By (a), these are

$$\text{cis } \frac{2\pi}{5}, \quad \text{cis } \frac{4\pi}{5}, \quad \text{cis } \left(-\frac{4\pi}{5}\right), \quad \text{cis } \left(-\frac{2\pi}{5}\right).$$

(d) (i) By De Moivre's theorem, $z^n + z^{-n} = \text{cis}(n\theta) + \text{cis}(-n\theta) = 2 \cos(n\theta)$. Therefore,

$$f(z) = (z^2 + z^{-2}) + (z + z^{-1}) + 1 = 2 \cos 2\theta + 2 \cos \theta + 1,$$

which is real.

(d) (ii) Using the double angle identity $\cos 2\theta = 2 \cos^2 \theta - 1$,

$$f(z) = 2(2 \cos^2 \theta - 1) + 2 \cos \theta + 1 = 4 \cos^2 \theta + 2 \cos \theta - 1.$$

(d) (iii) Writing $c = \cos \theta \in [-1, 1]$ and completing the square,

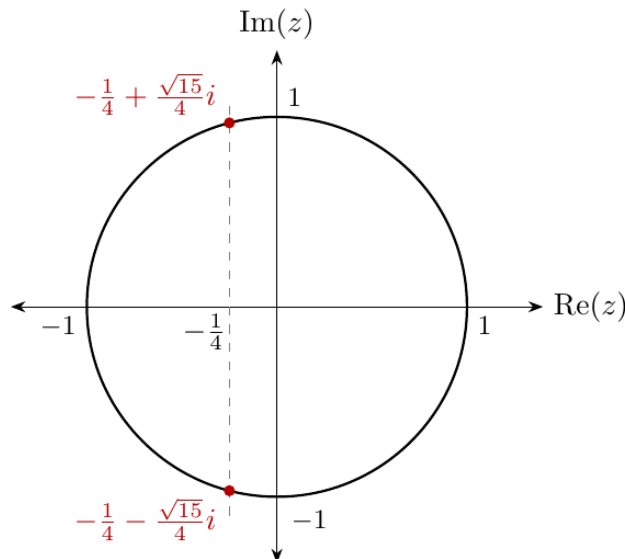
$$4c^2 + 2c - 1 = 4 \left(c + \frac{1}{4}\right)^2 - \frac{5}{4} \geq -\frac{5}{4},$$

with equality when $c = -\frac{1}{4} \in [-1, 1]$. Therefore, the minimum value of $f(z)$ is $-\frac{5}{4}$.

(d) (iv) The minimum occurs when $\cos \theta = -\frac{1}{4}$, i.e.

$$z = -\frac{1}{4} \pm \frac{\sqrt{15}}{4}i.$$

We mark these as follows.



Question 10. Parametric Curves in 3D

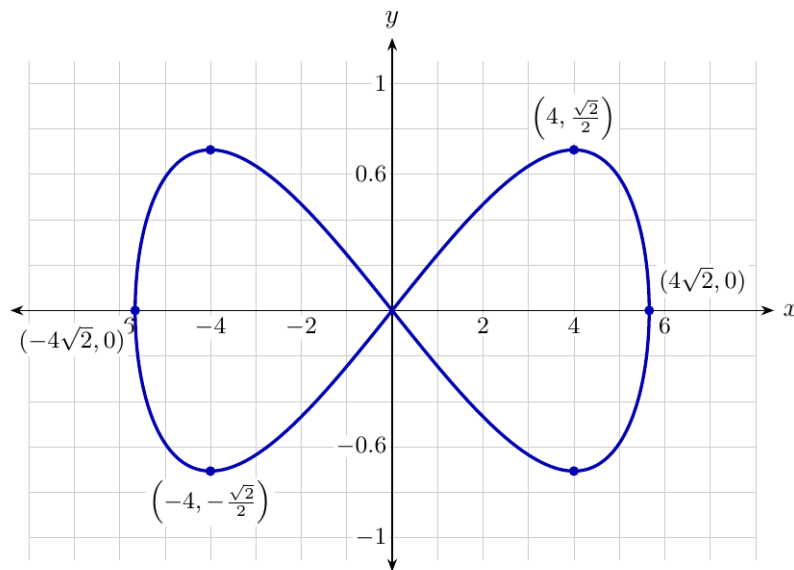
Solution. (a) Using the double angle identity $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$,

$$\int \cos^2 x \, dx = \int \frac{1}{2}(1 + \cos 2x) \, dx = \frac{1}{2}x + \frac{1}{4} \sin 2x + c.$$

(b) The curve is a figure of eight through the origin, with $|x| \leq 4\sqrt{2} \approx 5.66$ and $|y| \leq \frac{\sqrt{2}}{2} \approx 0.707$. The turning points $\left(\pm 4, \pm \frac{\sqrt{2}}{2}\right)$ occur at

$$t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}.$$

We sketch as follows.



(c) (i) Differentiating each component with respect to t ,

$$\mathbf{v} = \left[4\sqrt{2} \cos t, \sqrt{2} \cos 2t, 4 \right].$$

(c) (ii) Writing $c = \cos t$ and using $\cos 2t = 2c^2 - 1$,

$$\begin{aligned} |\mathbf{v}|^2 &= 32c^2 + 2(2c^2 - 1)^2 + 16 \\ &= 32c^2 + 8c^4 - 8c^2 + 2 + 16 \\ &= 8c^4 + 24c^2 + 18 \\ &= 2(2c^2 + 3)^2. \end{aligned}$$

Since $2c^2 + 3 > 0$, the speed is

$$|\mathbf{v}| = \sqrt{2}(2c^2 + 3) = 2\sqrt{2} \cos^2 t + 3\sqrt{2}.$$

(c) (iii) The distance travelled is the integral of the speed. By (a) and (c) (ii),

$$\begin{aligned}
 \int_{\pi/4}^{3\pi/4} (2\sqrt{2} \cos^2 t + 3\sqrt{2}) dt &= 2\sqrt{2} \left[\frac{t}{2} + \frac{1}{4} \sin 2t \right]_{\pi/4}^{3\pi/4} + 3\sqrt{2} \cdot \frac{\pi}{2} \\
 &= 2\sqrt{2} \left(\left(\frac{3\pi}{8} - \frac{1}{4} \right) - \left(\frac{\pi}{8} + \frac{1}{4} \right) \right) + \frac{3\sqrt{2}\pi}{2} \\
 &= 2\sqrt{2} \left(\frac{\pi}{4} - \frac{1}{2} \right) + \frac{3\sqrt{2}\pi}{2} \\
 &= \sqrt{2} \left(\frac{\pi}{2} - 1 + \frac{3\pi}{2} \right) \\
 &= \sqrt{2}(2\pi - 1).
 \end{aligned}$$

Remark 3. The z -component contributes a constant 4 to the velocity, so the dragonfly climbs steadily while tracing the figure of eight from (b) when viewed from above.

Remark 4. The diagrams have been drawn and the solutions have been verified with the help of Generative AI.

—Joel Kindiak, 17 Oct 26, 0022H