

2024 SACE Stage 2 Specialist Mathematics Examination Solutions

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Questions are publicly available. Do comment if you spot any errors.

Remark 1. In SACE Stage 2 Specialist Maths, the following equivalent notation is used:

$$\operatorname{cis} \theta \equiv \operatorname{cis}(\theta) \equiv \cos \theta + i \sin \theta.$$

Furthermore, for any planar shape S , we denote its area by $\mathcal{A}(S)$. Finally, we use row vector notation for vectors:

$$[x, y, z] \equiv \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Question 1. Polynomials

Solution. (a) Since P has real coefficients, $3 - i$ is also a zero by the conjugate root theorem. Therefore, a real quadratic factor is

$$(z - (3 + i))(z - (3 - i)) = (z - 3)^2 - i^2 = z^2 - 6z + 10.$$

(b) (i) By the factor theorem, $P(1) = 0$:

$$1 - 5 + a + b - 20 = 0 \quad \Rightarrow \quad a + b = 24.$$

(b) (ii) By the remainder theorem, $P(-1) = -34$:

$$1 + 5 + a - b - 20 = -34 \quad \Rightarrow \quad a - b = -20.$$

(b) (iii) Adding and subtracting the equations in (b) (i) and (b) (ii), $a = 2$ and $b = 22$.

(c) By (a) and (b),

$$P(z) = (z^2 - 6z + 10)(z^2 + pz - 2)$$

for some p , where we compared constant terms. Comparing the coefficients of z^3 , $p - 6 = -5$ so that $p = 1$. Therefore,

$$P(z) = (z^2 - 6z + 10)(z^2 + z - 2) = (z^2 - 6z + 10)(z - 1)(z + 2).$$

Question 2. Planes and Lines

Solution. (a) The normal vectors $[1, 2, -3]$ and $[1, 1, 2]$ of P_1 and P_2 are not scalar multiples of each other (comparing the first two components would require the scalar to be both 1 and 2). Therefore, the planes are not parallel.

(b) Applying $R_2 \rightarrow R_2 - R_1$,

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 1 & 1 & 2 & 5 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 0 & -1 & 5 & -4 \end{array} \right].$$

Setting $z = t$, the second row yields $-y + 5t = -4$, so that $y = 4 + 5t$. The first row then yields

$$x = 9 - 2(4 + 5t) + 3t = 1 - 7t.$$

Therefore, the planes meet along the line l_1 with parametric equations

$$x = 1 - 7t, y = 4 + 5t, z = t, \quad t \in \mathbb{R}.$$

(c) Setting $t = 2$, we obtain $x = 1 - 14 = -13$, $y = 4 + 10 = 14$ and $z = 2$. Hence, $A(-13, 14, 2)$ lies on l_1 .

(d) The plane P_3 contains the point $A(-13, 14, 2)$ on l_1 , the direction vector $[-7, 5, 1]$ of l_1 , and the vector

$$\overrightarrow{AQ} = [-10, 12, 0] - [-13, 14, 2] = [3, -2, -2].$$

Hence, a normal vector of P_3 is

$$\begin{aligned} [-7, 5, 1] \times [3, -2, -2] &= [5 \cdot (-2) - 1 \cdot (-2), 1 \cdot 3 - (-7) \cdot (-2), -7 \cdot (-2) - 5 \cdot 3] \\ &= [-8, -11, -1]. \end{aligned}$$

Taking the normal vector $[8, 11, 1]$ and the point A ,

$$P_3 : 8x + 11y + z = 8 \cdot (-13) + 11 \cdot 14 + 2 = -104 + 154 + 2 = 52.$$

As a check, Q satisfies

$$8 \cdot (-10) + 11 \cdot 12 + 0 = -80 + 132 = 52.$$

Question 3. Matrices and Mathematical Induction

Solution. (a) Denote $M = \begin{bmatrix} i & a \\ 0 & i \end{bmatrix}$. Using $i^2 = -1$,

$$M^2 = \begin{bmatrix} i^2 & ia + ai \\ 0 & i^2 \end{bmatrix} = \begin{bmatrix} -1 & 2ai \\ 0 & -1 \end{bmatrix}, \quad M^3 = M^2 M = \begin{bmatrix} -i & -a + 2ai^2 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} -i & -3a \\ 0 & -i \end{bmatrix}.$$

(b) For the base case $n = 1$, $\begin{bmatrix} i^1 & 1 \cdot ai^0 \\ 0 & i^1 \end{bmatrix} = \begin{bmatrix} i & a \\ 0 & i \end{bmatrix} = M$, as required. Fix $k \in \mathbb{N}^+$ such that

$$M^k = \begin{bmatrix} i^k & kai^{k-1} \\ 0 & i^k \end{bmatrix}.$$

Then

$$\begin{aligned} M^{k+1} &= M^k M = \begin{bmatrix} i^k & kai^{k-1} \\ 0 & i^k \end{bmatrix} \begin{bmatrix} i & a \\ 0 & i \end{bmatrix} \\ &= \begin{bmatrix} i^{k+1} & i^k a + kai^{k-1} \cdot i \\ 0 & i^{k+1} \end{bmatrix} \\ &= \begin{bmatrix} i^{k+1} & (k+1)ai^k \\ 0 & i^{k+1} \end{bmatrix}. \end{aligned}$$

Since the identity holds for $n = 1$ and it holds for $n = k + 1$ whenever it holds for $n = k$, the identity holds for all $n \in \mathbb{N}^+$ by mathematical induction.

(c) Since $2027 = 4 \cdot 506 + 3$ and $i^4 = 1$, $i^{2027} = i^3 = -i$ and $i^{2026} = i^2 = -1$. By (b),

$$M^{2027} = \begin{bmatrix} -i & -2027a \\ 0 & -i \end{bmatrix}.$$

Therefore, by (a),

$$M^3 - M^{2027} = \begin{bmatrix} -i & -3a \\ 0 & -i \end{bmatrix} - \begin{bmatrix} -i & -2027a \\ 0 & -i \end{bmatrix} = \begin{bmatrix} 0 & 2024a \\ 0 & 0 \end{bmatrix}.$$

Question 4. Vector Geometry

Solution. (a) (i) Since N is the midpoint of OB , $\overrightarrow{AN} = \overrightarrow{ON} - \overrightarrow{OA} = \frac{1}{2}\mathbf{b} - \mathbf{a}$.

(a) (ii) Similarly, $\overrightarrow{BM} = \overrightarrow{OM} - \overrightarrow{OB} = \frac{1}{2}\mathbf{a} - \mathbf{b}$.

(b) Since $\overrightarrow{AN} \perp \overrightarrow{BM}$, their dot product is zero:

$$\begin{aligned} 0 &= \left(\frac{1}{2}\mathbf{b} - \mathbf{a}\right) \cdot \left(\frac{1}{2}\mathbf{a} - \mathbf{b}\right) \\ &= \frac{1}{4}\mathbf{a} \cdot \mathbf{b} - \frac{1}{2}|\mathbf{b}|^2 - \frac{1}{2}|\mathbf{a}|^2 + \mathbf{a} \cdot \mathbf{b} \\ &= \frac{5}{4}\mathbf{a} \cdot \mathbf{b} - \frac{1}{2}(|\mathbf{a}|^2 + |\mathbf{b}|^2). \end{aligned}$$

Rearranging, $|\mathbf{a}|^2 + |\mathbf{b}|^2 = \frac{5}{2}\mathbf{a} \cdot \mathbf{b}$.

(c) Let $\theta = \angle AOB$. By the cosine rule in triangle OAB ,

$$|\overrightarrow{AB}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\mathbf{a} \cdot \mathbf{b},$$

since $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$ by definition.

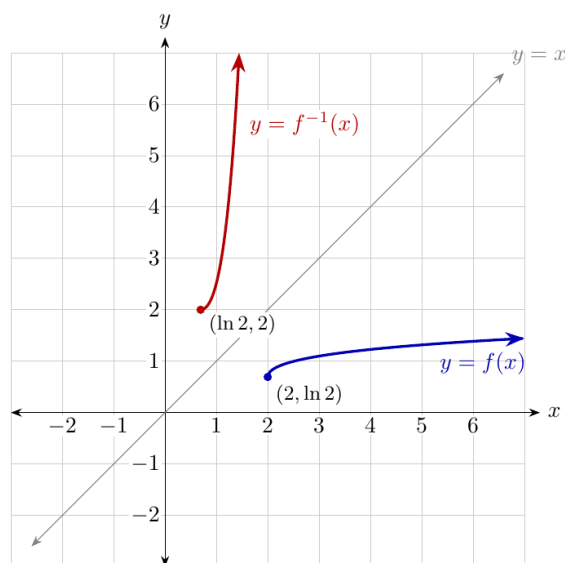
(d) By (b), $2\mathbf{a} \cdot \mathbf{b} = \frac{4}{5}(|\mathbf{a}|^2 + |\mathbf{b}|^2)$. Substituting into (c),

$$|\overrightarrow{AB}|^2 = (|\mathbf{a}|^2 + |\mathbf{b}|^2) - \frac{4}{5}(|\mathbf{a}|^2 + |\mathbf{b}|^2) = \frac{1}{5}(|\mathbf{a}|^2 + |\mathbf{b}|^2).$$

Remark 2. Equivalently, $|OA|^2 + |OB|^2 = 5|AB|^2$: whenever two medians of a triangle are perpendicular, the squares of the two sides they bisect sum to five times the square of the third side.

Question 5. Inverse Functions and Volumes

Solution. (a) The curve starts at $(2, \ln 2) \approx (2, 0.693)$ and increases slowly, e.g. through $(6, \ln 4) \approx (6, 1.39)$. We sketch it in blue below, together with $y = f^{-1}(x)$ in red for (b) (iii).



(b) (i) Let $y = \ln(\sqrt{x-2} + 2)$. Then

$$e^y = \sqrt{x-2} + 2 \Rightarrow \sqrt{x-2} = e^y - 2 \Rightarrow x = (e^y - 2)^2 + 2,$$

where squaring is reversible since $e^y - 2 = \sqrt{x-2} \geq 0$. Swapping x and y ,

$$f^{-1}(x) = (e^x - 2)^2 + 2.$$

(b) (ii) The domain of f^{-1} is the range of f . Since f is increasing with $f(2) = \ln 2$ and $f(x) \rightarrow \infty$ as $x \rightarrow \infty$, the domain of f^{-1} is $[\ln 2, \infty)$.

(b) (iii) The graph of $y = f^{-1}(x)$ is the reflection of $y = f(x)$ in the line $y = x$. It starts at $(\ln 2, 2)$ and increases steeply; see the red curve in the figure in (a).

(c) (i) The integral

$$\pi \int_2^5 (\ln(\sqrt{y-2} + 2))^2 dy$$

only depends on the integrand and the limits, not on the name of the variable. Replacing y by x gives the volume formed when $y = f(x)$ is rotated about the x -axis from $x = 2$ to $x = 5$. Geometrically, by Figure 3, reflecting the region in $y = x$ swaps the two solids, so their volumes are equal.

(c) (ii) Using software,

$$V = \pi \int_2^5 (\ln(\sqrt{x-2} + 2))^2 dx \approx 12.4 \text{ units}^3.$$

Question 6. Complex Roots

Solution. (a) (i) Let $z = r \operatorname{cis} \theta$ with $r > 0$ and $-\pi < \theta \leq \pi$. By De Moivre's theorem,

$$r^6 \operatorname{cis} 6\theta = 8 \operatorname{cis}((2k+1)\pi),$$

so that $r = 8^{1/6} = \sqrt{2}$ and $\theta = \frac{(2k+1)\pi}{6}$. Matching the positions in Figure 4,

$$\begin{aligned} z_1 &= \sqrt{2} \operatorname{cis} \frac{\pi}{6}, & z_2 &= \sqrt{2} \operatorname{cis} \frac{\pi}{2}, & z_3 &= \sqrt{2} \operatorname{cis} \frac{5\pi}{6}, \\ z_4 &= \sqrt{2} \operatorname{cis} \left(-\frac{5\pi}{6}\right), & z_5 &= \sqrt{2} \operatorname{cis} \left(-\frac{\pi}{2}\right), & z_6 &= \sqrt{2} \operatorname{cis} \left(-\frac{\pi}{6}\right). \end{aligned}$$

(a) (ii) Since z_6 is the conjugate of z_1 ,

$$z_1 - z_6 = z_1 - \bar{z}_1 = 2i \operatorname{Im}(z_1) = 2i \cdot \sqrt{2} \sin \frac{\pi}{6} = \sqrt{2} i,$$

so that $|z_1 - z_6| = \sqrt{2}$.

(b) (i) The triangle Oz_1z_6 has two sides of length $\sqrt{2}$ with included angle $\frac{\pi}{6} - (-\frac{\pi}{6}) = \frac{\pi}{3}$. Hence,

$$\mathcal{A}(\Delta Oz_1z_6) = \frac{1}{2} \cdot \sqrt{2} \cdot \sqrt{2} \cdot \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}.$$

(b) (ii) By symmetry, the hexagon consists of six such triangles, so its area is $6 \cdot \frac{\sqrt{3}}{2} = 3\sqrt{3}$.

(c) The solutions of $z^6 = k$ lie on a circle of radius $R = |k|^{1/6}$, spaced $\frac{\pi}{3}$ apart. As in (b), the hexagon has area

$$6 \cdot \frac{1}{2} R^2 \sin \frac{\pi}{3} = \frac{3\sqrt{3}}{2} R^2 = 9\sqrt{3} \quad \Rightarrow \quad R^2 = 6 \quad \Rightarrow \quad |k| = R^6 = 216.$$

Therefore, $k = -216$ works (and so does $k = 216$).

Question 7. Integration by Parts

Solution. (a) Integrating by parts with

$$u = \ln(\cos x), \quad dv = \sin x \, dx, \quad du = -\frac{\sin x}{\cos x} \, dx, \quad v = -\cos x,$$

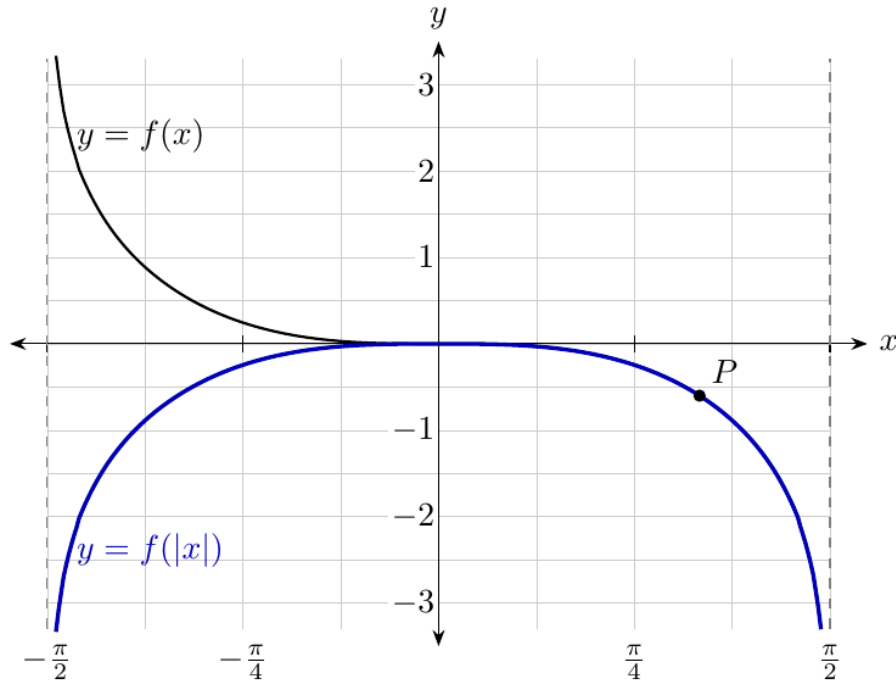
we obtain

$$\begin{aligned} \int \sin x \ln(\cos x) \, dx &= -\cos x \ln(\cos x) - \int (-\cos x) \cdot \left(-\frac{\sin x}{\cos x}\right) \, dx \\ &= -\cos x \ln(\cos x) - \int \sin x \, dx \\ &= -\cos x \ln(\cos x) + \cos x + c. \end{aligned}$$

(b) By (a),

$$\begin{aligned} \int_0^{\pi/3} \sin x \ln(\cos x) \, dx &= [-\cos x \ln(\cos x) + \cos x]_0^{\pi/3} \\ &= \left(-\frac{1}{2} \ln \frac{1}{2} + \frac{1}{2}\right) - (-1 \cdot \ln 1 + 1) \\ &= \frac{1}{2} \ln 2 + \frac{1}{2} - 1 \\ &= \frac{1}{2} \ln 2 - \frac{1}{2}. \end{aligned}$$

(c) For $x \geq 0$, $f(|x|) = f(x)$. For $x < 0$, the graph of $y = f(|x|)$ is the reflection of the right half of $y = f(x)$ in the y -axis. We sketch as follows.



(d) Since f is odd,

$$f(|x|) = f(-x) = -f(x),$$

for $x < 0$, and the two curves coincide for $x \geq 0$. Hence, the bounded region lies in $-\frac{\pi}{3} \leq x \leq 0$, where

$$f(x) \geq 0 \geq f(|x|).$$

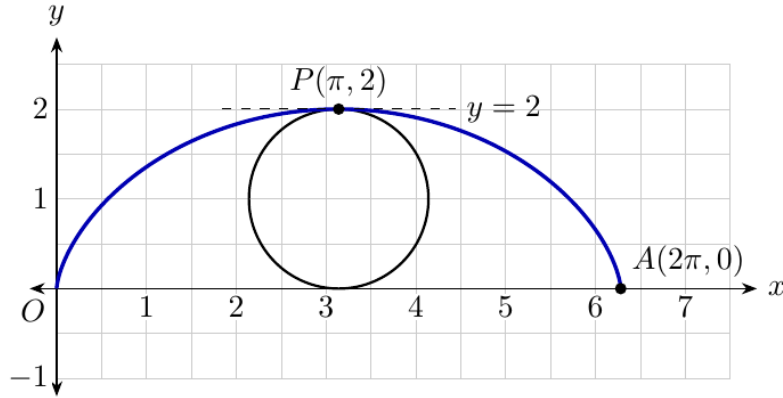
Its area is

$$\begin{aligned} \int_{-\pi/3}^0 (f(x) - f(|x|)) \, dx &= 2 \int_{-\pi/3}^0 f(x) \, dx \\ &= -2 \int_0^{\pi/3} f(x) \, dx \\ &= -2 \left(\frac{1}{2} \ln 2 - \frac{1}{2} \right) \\ &= 1 - \ln 2 \text{ units}^2, \end{aligned}$$

where we used (b) and the substitution $x \mapsto -x$ together with $f(-x) = -f(x)$.

Question 8. Parametric Curves

Solution. (a) The curve is a cycloid: it starts at $O(0, 0)$, rises to its maximum $(\pi, 2)$ at $t = \pi$, and returns to $(2\pi, 0)$ at $t = 2\pi$. We sketch as follows.



(b) (i) Differentiating $(x - \pi)^2 + (y - 1)^2 = 1$ implicitly with respect to x ,

$$2(x - \pi) + 2(y - 1) \frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{x - \pi}{y - 1}.$$

(b) (ii) By the chain rule,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sin t}{1 - \cos t}.$$

(c) The point $P(\pi, 2)$ lies on the circle since $0^2 + 1^2 = 1$, and on the parametric curve with $t = \pi$. By (b) (i), the circle has gradient $-\frac{0}{1} = 0$ at P . By (b) (ii), the parametric curve has gradient

$$\frac{\sin \pi}{1 - \cos \pi} = \frac{0}{2} = 0$$

at P . Therefore, both curves have the common horizontal tangent $y = 2$ at P .

(d) By symmetry of the curve about $x = \pi$, the arc PA has the same length as the arc OP , and the length of each arc is greater than the length of the corresponding chord. By the triangle inequality in triangle OPA ,

$$\text{length of curve} = \text{arc } OP + \text{arc } PA > |OP| + |PA| > |OA| = 2\pi.$$

Since the circle has radius 1, its circumference is 2π . Hence, the length of the parametric curve is greater than the circumference of the circle.

(e) (i) Since $x'(t) = 1 - \cos t$ and $y'(t) = \sin t$,

$$(x'(t))^2 + (y'(t))^2 = 1 - 2\cos t + \cos^2 t + \sin^2 t = 2 - 2\cos t = 4\sin^2 \frac{t}{2},$$

using the double angle identity $\cos t = 1 - 2\sin^2 \frac{t}{2}$. For $0 \leq t \leq 2\pi$, $\sin \frac{t}{2} \geq 0$, so that the length of the curve is

$$\int_0^{2\pi} \sqrt{(x'(t))^2 + (y'(t))^2} dt = \int_0^{2\pi} 2\sin \left(\frac{t}{2} \right) dt.$$

(e) (ii) By (e) (i),

$$\int_0^{2\pi} 2\sin \left(\frac{t}{2} \right) dt = \left[-4\cos \frac{t}{2} \right]_0^{2\pi} = -4(-1) + 4 = 8 \text{ units.}$$

Remark 3. This confirms (d): $8 > 2\pi \approx 6.28$. In fact, the bound in (d) is $2\sqrt{\pi^2 + 4} \approx 7.45$, since $|OP| = |PA| = \sqrt{\pi^2 + 4}$.

Question 9. Differential Equations

Solution. (a) (i) By the method of separable variables,

$$\begin{aligned}\int y^{-2} dy &= \int -\frac{1}{10} e^{-0.5t} dt \\ -\frac{1}{y} &= \frac{1}{5} e^{-0.5t} + C \\ \frac{1}{y} &= c - 0.2e^{-0.5t} \\ y &= \frac{1}{c - 0.2e^{-0.5t}},\end{aligned}$$

where we set $c := -C$.

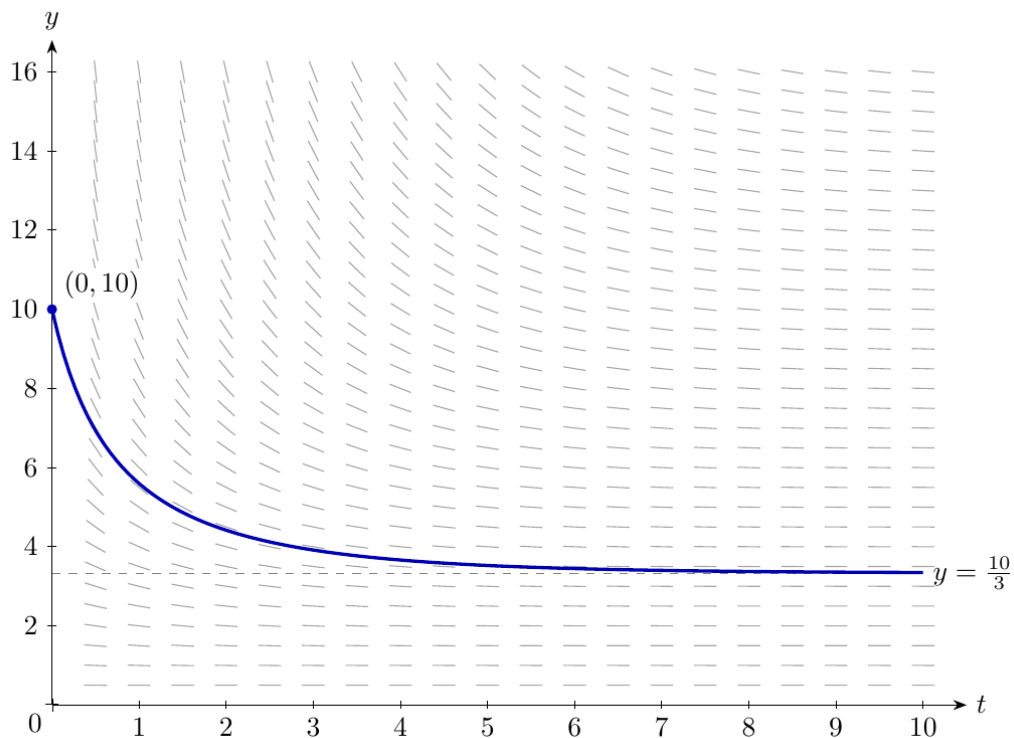
(a) (ii) Setting $t = 0$ and $y = 10$,

$$10 = \frac{1}{c - 0.2} \quad \Rightarrow \quad c = 0.3.$$

As $t \rightarrow \infty$, $e^{-0.5t} \rightarrow 0$, so that

$$y = \frac{1}{0.3 - 0.2e^{-0.5t}} \rightarrow \frac{1}{0.3} = \frac{10}{3}.$$

(b) The solution curve starts at $(0, 10)$, decreases, and approaches $y = \frac{10}{3}$ from above. We sketch as follows.



(c) Differentiating $\frac{dy}{dt} = -\frac{1}{10}e^{-0.5t}y^2$ with respect to t using the product rule and the chain rule,

$$\begin{aligned}\frac{d^2y}{dt^2} &= -\frac{1}{10} \left(-\frac{1}{2}e^{-0.5t}y^2 + e^{-0.5t} \cdot 2y \frac{dy}{dt} \right) \\ &= -\frac{1}{2} \cdot \left(-\frac{1}{10}e^{-0.5t}y^2 \right) - \frac{1}{5}e^{-0.5t}y \frac{dy}{dt} \\ &= -\frac{1}{2} \frac{dy}{dt} - \frac{1}{5}e^{-0.5t}y \frac{dy}{dt} \\ &= -\frac{dy}{dt} \left(\frac{1}{2} + \frac{1}{5}e^{-0.5t}y \right).\end{aligned}$$

(d) Since $\frac{dy}{dt} \neq 0$ (as $y > 0$), by (c),

$$-\left(\frac{1}{2} + \frac{1}{5}e^{-0.5t}y\right) = -1 \quad \Rightarrow \quad e^{-0.5t}y = \frac{5}{2}.$$

Write $u = e^{-0.5t}$. By (a) (ii),

$$y = \frac{1}{0.3 - 0.2u},$$

so that

$$\frac{u}{0.3 - 0.2u} = \frac{5}{2} \quad \Rightarrow \quad u = 0.75 - 0.5u \quad \Rightarrow \quad u = \frac{1}{2}.$$

Therefore, $e^{-0.5t} = \frac{1}{2}$, so that $t = 2 \ln 2$.

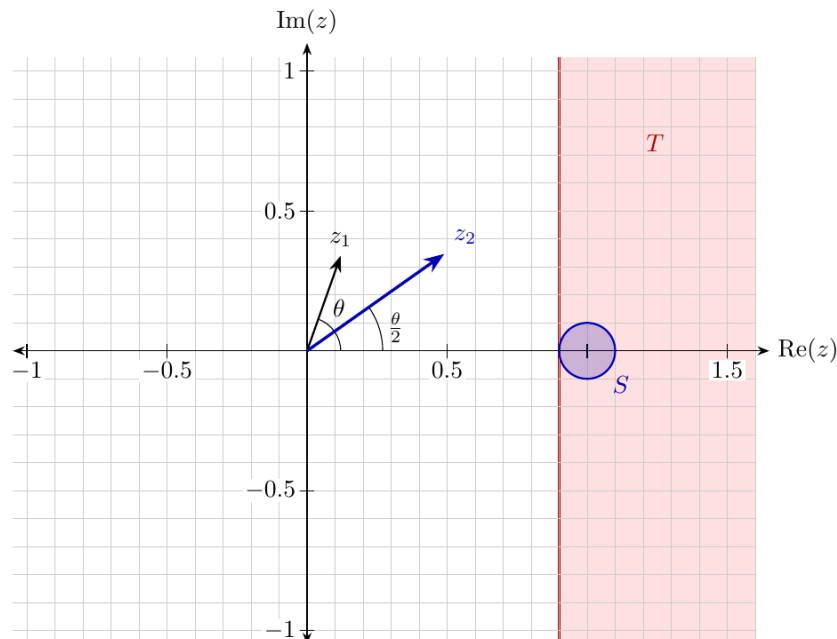
Question 10. Complex Roots and Regions

Solution. (a) (i) By De Moivre's theorem, $z_2 = \sqrt{0.36} \operatorname{cis} \frac{\theta}{2} = 0.6 \operatorname{cis} \frac{\theta}{2}$ satisfies $z_2^2 = 0.36 \operatorname{cis} \theta = z_1$. Since $0 < \frac{\theta}{2} < \frac{\pi}{4}$, z_2 is in the first quadrant.

(a) (ii) The point z_2 has modulus 0.6 and half the argument of z_1 , shown in blue in the figure below.

(b) (i) The set S is the closed disc of radius 0.1 centred at 1, shaded in blue below.

(b) (ii) The set T is the closed half-plane on and to the right of the line $\operatorname{Re}(z) = 0.9$, shaded in red below.



(c) (i) Since $\ln 0.9 < 0$,

$$\begin{aligned} (0.36)^{\frac{1}{n}} \geq 0.9 &\Leftrightarrow \frac{1}{n} \ln 0.36 \geq \ln 0.9 \\ &\Leftrightarrow n \geq \frac{\ln 0.36}{\ln 0.9} \approx 9.70, \end{aligned}$$

where we multiplied by $n / \ln 0.9 < 0$. Therefore, the smallest such integer is $n = 10$.

(c) (ii) By De Moivre's theorem, $w = (0.36)^{\frac{1}{10}} \operatorname{cis} \frac{\theta}{10}$, which is in the first quadrant since $0 < \frac{\theta}{10} < \frac{\pi}{20}$.

(c) (iii) (1) Setting $\theta = \frac{\pi}{5}$, $w = (0.36)^{0.1} \operatorname{cis} \frac{\pi}{50}$. In cartesian form,

$$w = (0.36)^{0.1} \cos \frac{\pi}{50} + i (0.36)^{0.1} \sin \frac{\pi}{50} \approx 0.901 + 0.0567i.$$

(c) (iii) (2) Yes. Since $\operatorname{Re}(w) \approx 0.901 \geq 0.9$, w is in T . (This is guaranteed by (c) (i), since $\operatorname{Re}(w) = |w| \cos \frac{\pi}{50}$ with $|w| = 0.903$.)

(c) (iii) (3) No. We compute

$$|w - 1| \approx |-0.0989 + 0.0567i| = \sqrt{0.0989^2 + 0.0567^2} \approx 0.114 > 0.1,$$

so that w is not within S .

Remark 4. In particular, S is a proper subset of T , and w lies in $T \setminus S$: it is close enough to the real axis to satisfy $\operatorname{Re}(w) \geq 0.9$, but its imaginary part is too large for it to lie within 0.1 of 1.

Remark 5. The diagrams have been drawn and the solutions have been verified with the help of Generative AI.

—Joel Kindiak, 16 Oct 26, 2344H