

2025 SACE Stage 2 Specialist Mathematics Examination Solutions

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Questions are publicly [available](#). Do comment if you spot any errors.

Remark 1. In SACE Stage 2 Specialist Maths, the following equivalent notation is used:

$$\operatorname{cis} \theta \equiv \operatorname{cis}(\theta) \equiv \cos \theta + i \sin \theta.$$

Furthermore, for any planar shape S , we denote its area by $\mathcal{A}(S)$. Finally, we use row vector notation for vectors:

$$[x, y, z] \equiv \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

and denote vector norms by $|\mathbf{v}| \equiv \|\mathbf{v}\|$.

Question 1. Related Rates of Change

Solution. (a) Setting $V = \frac{200}{3}\pi$,

$$\frac{11\pi}{2}h^2 - \frac{\pi}{3}h^3 = \frac{200\pi}{3} \quad \Rightarrow \quad 2h^3 - 33h^2 + 400 = 0.$$

Solving for $0 \leq h \leq 5$, we have $h = 4$.

(b) Differentiating with respect to h ,

$$\begin{aligned} \frac{dV}{dh} &= \frac{d}{dh} \left(\frac{11\pi}{2}h^2 - \frac{\pi}{3}h^3 \right) \\ &= \frac{11\pi}{2} \frac{d}{dh} (h^2) - \frac{\pi}{3} \cdot \frac{d}{dh} (h^3) \\ &= \frac{11\pi}{2} \cdot 2h - \frac{\pi}{3} \cdot 3h^2 \\ &= \pi h(11 - h). \end{aligned}$$

By the chain rule,

$$\frac{dV}{dt} = \pi h(11 - h) \cdot \frac{dh}{dt} \quad \Rightarrow \quad \frac{dh}{dt} = \frac{1}{\pi h(11 - h)} \frac{dV}{dt}.$$

(c) By (a), $h = 4$ when $V = \frac{200}{3}\pi$. By the question, $\frac{dV}{dt} = 2$. By (b),

$$\frac{dh}{dt} = \frac{1}{\pi \cdot 4 \cdot (11 - 4)} \cdot 2 = \frac{1}{14\pi} \approx 0.0227 \text{ cm/s}.$$

Question 2. Integral Calculus

Solution. (a) Integrating by parts and using the assumed integral,

$$\begin{aligned}\int x^2 e^{2x} dx &= \int x \cdot x e^{2x} dx \\&= \underbrace{x}_u \cdot \underbrace{\left(\frac{1}{2}x e^{2x} - \frac{1}{4}e^{2x}\right)}_v - \int \underbrace{\left(\frac{1}{2}x e^{2x} - \frac{1}{4}e^{2x}\right)}_v \cdot \underbrace{1}_{du} dx \\&= \frac{1}{2}x^2 e^{2x} - \frac{1}{4}x e^{2x} - \left(\frac{1}{2} \int x e^{2x} dx - \frac{1}{4} \int e^{2x} dx\right) \\&= \frac{1}{2}x^2 e^{2x} - \frac{1}{4}x e^{2x} - \frac{1}{2} \int x e^{2x} dx + \frac{1}{4} \int e^{2x} dx \\&= \frac{1}{2}x^2 e^{2x} - \frac{1}{4}x e^{2x} - \frac{1}{2} \cdot \left(\frac{1}{2}x e^{2x} - \frac{1}{4}e^{2x}\right) + \frac{1}{4} \cdot \frac{1}{2}e^{2x} + C \\&= \frac{1}{2}x^2 e^{2x} - \frac{1}{2}x e^{2x} + \frac{1}{4}e^{2x} + C.\end{aligned}$$

(b) Using the volume of revolution formula, the required volume V is given by

$$\begin{aligned}V &= \pi \int_{-1}^0 (x e^x)^2 dx \\&= \pi \int_{-1}^0 x^2 e^{2x} dx \\&= \pi \left[\frac{1}{2}x^2 e^{2x} - \frac{1}{2}x e^{2x} + \frac{1}{4}e^{2x} \right]_{-1}^0 \\&= \frac{1}{4}\pi(1 - e^{-2}) \text{ units}^3.\end{aligned}$$

Question 3. Complex Number Algebra

Solution. (a) (i) By Euler's identity and the 2π -periodicity of cosine and sine,

$$\operatorname{cis} 3\pi = \operatorname{cis} \pi = -1.$$

By De Moivre's theorem,

$$\begin{aligned}\left(2 \operatorname{cis} \frac{3\pi}{5}\right)^5 &= 2^5 \operatorname{cis} \left(5 \cdot \frac{3\pi}{5}\right) \\&= 32 \operatorname{cis} 3\pi \\&= 32 \cdot (-1) \\&= -32.\end{aligned}$$

(a) (ii) Let $z = r \operatorname{cis} \theta$ denote a solution to $z^5 = -32$, where $r > 0$ and $-\pi < \theta \leq \pi$. By De Moivre's theorem,

$$r^5 \operatorname{cis} 5\theta = z^5 = -32 = 32 \operatorname{cis}((2k+1)\pi).$$

Comparing moduli, $r^5 = 32 \Rightarrow r = 2$. Comparing arguments,

$$\theta = \frac{2k+1}{5}\pi \in (-\pi, \pi].$$

Solving for k , we get

$$-5 < 2k+1 \leq 5,$$

so that $k = -2, -1, 0, 1, 2$. Therefore, we obtain the solutions

$$-2, \quad 2 \operatorname{cis} \frac{\pi}{5}, \quad 2 \operatorname{cis} \frac{3\pi}{5}, \quad 2 \operatorname{cis} \left(-\frac{3\pi}{5}\right), \quad 2 \operatorname{cis} \left(-\frac{\pi}{5}\right)$$

from $k = 2, 0, 1, -2, -1$ respectively.

(b) (i) We take advantage of the geometric series:

$$\begin{aligned} \frac{z^5 + 32}{z + 2} &= \frac{-32}{-2} \cdot \frac{(-z/2)^5 - 1}{(-z/2) - 1} \\ &= 16 \cdot (1 + (-z/2) + (-z/2)^2 + (-z/2)^3 + (-z/2)^4) \\ &= 16 \cdot \left(1 - \frac{z}{2} + \frac{z^2}{4} - \frac{z^3}{8} + \frac{z^4}{16}\right) \\ &= 16 - 8z + 4z^2 - 2z^3 + z^4, \end{aligned}$$

as required, where the final result follows from the commutativity of addition.

(b) (ii) Using (a) (ii), since $z \neq -2$, we obtain the solutions

$$2 \operatorname{cis} \frac{\pi}{5}, \quad 2 \operatorname{cis} \frac{3\pi}{5}, \quad 2 \operatorname{cis} \left(-\frac{3\pi}{5}\right), \quad 2 \operatorname{cis} \left(-\frac{\pi}{5}\right).$$

(c) (i) Using the formula for the area of a triangle,

$$\mathcal{A}(\triangle AOD) = \frac{1}{2} \cdot 2 \cdot 2 \sin \frac{2\pi}{5} = 2 \sin \frac{2\pi}{5}.$$

(c) (ii) Similarly,

$$\mathcal{A}(\triangle BOC) = \frac{1}{2} \cdot 2 \cdot 2 \sin \frac{4\pi}{5} = 2 \sin \frac{4\pi}{5}.$$

By symmetry,

$$\mathcal{A}(ABCD) = 6 \sin \frac{2\pi}{5} + 2 \sin \frac{4\pi}{5}.$$

Therefore, $(a, b) = (6, 2)$.

Question 4. Mathematical Induction

Solution. (a) By the product rule,

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}((2x+1)e^x) \\ &= 2e^x + (2x+1)e^x \\ &= (2+2x+1)e^x \\ &= (2x+3)e^x. \end{aligned}$$

(b) For the base case $n = 1$, part (a) yields

$$\frac{dy}{dx} = (2x+3)e^x = (2x + (2 \cdot 1 + 1))e^x,$$

as required. Fix $k \in \mathbb{N}^+$ such that

$$\frac{d^k y}{dx^k} = (2x + (2k+1))e^x.$$

By the product rule,

$$\begin{aligned}
 \frac{d^{k+1}y}{dx^{k+1}} &= \frac{d}{dx}((2x + (2k + 1))e^x) \\
 &= 2e^x + (2x + (2k + 1))e^x \\
 &= (2 + 2x + (2k + 1))e^x \\
 &= (2x + 2k + 3)e^x \\
 &= (2x + 2(k + 1) + 1)e^x.
 \end{aligned}$$

Since the identity holds for $n = 1$ and it holds for $n = k + 1$ whenever $n = k$, the identity holds for $n \in \mathbb{N}^+$ by mathematical induction.

Question 5. Vector Algebra

Solution. (a) Using cross product properties,

$$\begin{aligned}
 (2\mathbf{a} + 3\mathbf{b}) \times (\mathbf{a} + 5\mathbf{b}) &= 2\mathbf{a} \times \mathbf{a} + 10\mathbf{a} \times \mathbf{b} + 3\mathbf{b} \times \mathbf{a} + 15\mathbf{b} \times \mathbf{b} \\
 &= \mathbf{0} + 10\mathbf{a} \times \mathbf{b} - 3\mathbf{a} \times \mathbf{b} + \mathbf{0} \\
 &= 7\mathbf{a} \times \mathbf{b}.
 \end{aligned}$$

(b) (i) Using vector addition,

$$\begin{aligned}
 \overrightarrow{OE} &= \overrightarrow{OC} + \overrightarrow{CD} + \overrightarrow{DE} \\
 &= \mathbf{a} + \mathbf{a} + 3\mathbf{b} \\
 &= 2\mathbf{a} + 3\mathbf{b}.
 \end{aligned}$$

(b) (ii) Similarly,

$$\overrightarrow{OF} = \mathbf{a} + 5\mathbf{b}.$$

By (a),

$$\begin{aligned}
 \mathcal{A}(\triangle OEF) &= \frac{1}{2}|\overrightarrow{OE} \times \overrightarrow{OF}| \\
 &= \frac{1}{2}|(2\mathbf{a} + 3\mathbf{b}) \times (\mathbf{a} + 5\mathbf{b})| \\
 &= \frac{1}{2}|7\mathbf{a} \times \mathbf{b}| \\
 &= \frac{7}{2}|\mathbf{a} \times \mathbf{b}|.
 \end{aligned}$$

(b) (iii) Similarly,

$$\begin{aligned}
 \mathcal{A}(\triangle ODE) &= \frac{1}{2}|\overrightarrow{OD} \times \overrightarrow{DE}| \\
 &= \frac{1}{2}|2\mathbf{a} \times 3\mathbf{b}| \\
 &= 3|\mathbf{a} \times \mathbf{b}|.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \mathcal{A}(ODEF) &= \mathcal{A}(\triangle OEF) + \mathcal{A}(\triangle ODE) \\
 &= \frac{7}{2}|\mathbf{a} \times \mathbf{b}| + 3|\mathbf{a} \times \mathbf{b}| \\
 &= \frac{13}{2}|\mathbf{a} \times \mathbf{b}|.
 \end{aligned}$$

(c) Computing the cross product,

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= [1, 2, 4] \times [3, 1, -5] \\ &= [-10 - 4, -5 - 12, 1 - 6] \\ &= [-14, -17, -5] \\ &= -[14, 17, 5].\end{aligned}$$

Computing the vector norm,

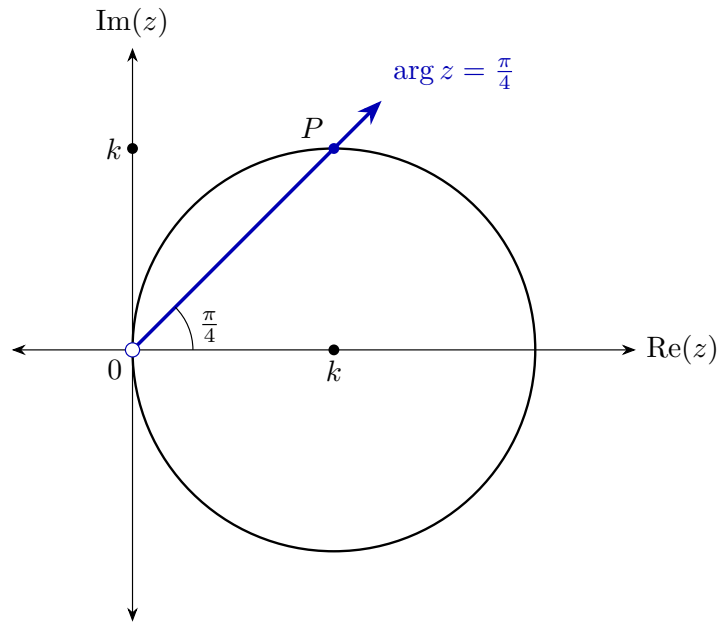
$$|\mathbf{a} \times \mathbf{b}| = \sqrt{14^2 + 17^2 + 5^2} = \sqrt{510}.$$

By (b) (iii),

$$\mathcal{A}(ODEF) = \frac{13}{2} \sqrt{510} \approx 147 \text{ units}^2.$$

Question 6. Complex Number Geometry

Solution. (a) See the figure below.



(b) (i) In Cartesian form, $P = k + ki$.

(b) (ii) In polar form, $P = k\sqrt{2} \operatorname{cis} \frac{1}{4}\pi$.

(c) $C_n : |z - 2^n| = 2^n$.

(d) We have

$$|v - 2^2| = |1 + 3i| = \sqrt{10} < \sqrt{16} = 2^2,$$

so that v lies inside C_2 .

(e) Since w has argument $\pi/4$ it lies on the ray drawn in (a). For any point z on the ray, z lies on C_n if and only if $|z| = 2^{n+1/2}$. Hence, we aim to find the smallest n such that

$$|w| < 2^{n+1/2}.$$

Since $|w| = 10$, taking logarithms yields

$$n > \frac{\ln 10}{\ln 2} - 1/2 \approx 2.82.$$

Therefore, $n = 3$.

Question 7. Functions

Solution. (a) (i) Using the Pythagorean identity,

$$\begin{aligned}\sin^3 x &= \sin x \cdot \sin^2 x \\ &= \sin x \cdot (1 - \cos^2 x) \\ &= \sin x - \sin x \cos^2 x.\end{aligned}$$

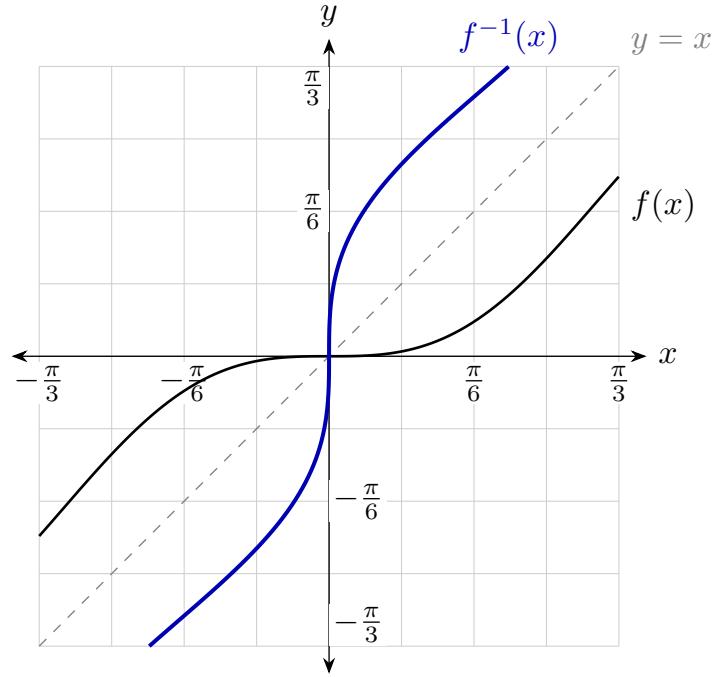
(a) (ii) Using linearity and the reverse chain rule,

$$\begin{aligned}\int \sin^3 x \, dx &= \int \sin x \, dx + \int (-\sin x)(\cos x)^2 \, dx \\ &= -\cos x + \frac{(\cos x)^3}{3} + C \\ &= -\cos x + \frac{1}{3} \cos^3 x + C.\end{aligned}$$

Substituting the limits of the definite integral,

$$\begin{aligned}\int_0^{\pi/3} \sin^3 x \, dx &= \left[-\cos x + \frac{1}{3} \cos^3 x \right]_0^{\pi/3} \\ &= \left(-\cos(\pi/3) + \frac{1}{3} \cos^3(\pi/3) \right) - \left(-1 + \frac{1}{3} \cdot 1 \right) \\ &= \left(-\frac{1}{2} + \frac{1}{3} \cdot \left(\frac{1}{2} \right)^3 \right) + \frac{2}{3} \\ &= \frac{5}{24}.\end{aligned}$$

(b) See the figure below.



(c) $g(x) = f(|x|)$.

(d) The area of the white space, by (a) (ii), is $5/24$. By symmetry, the total area of the white space is $8 \times 5/24 = 5/3$. Since the square has an area of $(2\pi/3)^2 = 4\pi^2/9$, the required area is what is left over:

$$\left(\frac{4\pi^2}{9} - \frac{5}{3} \right) \text{ units}^2.$$

Question 8. Vector Geometry

Solution. (a) (i) Substituting $x = d_1/a, y = 0, z = 0$, we get

$$a \cdot \frac{d_1}{a} + b \cdot 0 + c \cdot 0 = d_1,$$

as required.

(a) (ii) Similarly, the point $(d_2/a, 0, 0)$ lies on the plane

$$ax + by + cz - d_2 = 0.$$

By (a) (i), the distance between the planes is the distance from $(d_1/a, 0, 0)$ to the second plane:

$$\frac{|a \cdot d_1/a + b \cdot 0 + c \cdot 0 - d_2|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}.$$

(b) (i) Particularising (a) (ii) to $(a, b, c, d_1, d_2) = (2, -1, 1, 20, -10)$, we obtain a distance of

$$\frac{|20 - (-10)|}{\sqrt{2^2 + (-1)^2 + 1^2}} = \frac{30}{\sqrt{6}} = 5\sqrt{6} \approx 12.2 \text{ units}.$$

(b) (ii) Since the normal vector of P_1 is given by $[2, -1, 1]$, the normal is given by the line equation

$$[x, y, z] = [10, 2, 2] + t[2, -1, 1], \quad t \in \mathbb{R}.$$

(b) (iii) Writing in parametric form,

$$\overrightarrow{OB} = [10 + 2t, 2 - t, 2 + t]$$

for some t . When the normal intersects P_2 at B ,

$$2(10 + 2t) - (2 - t) + (2 + t) = -10 \Rightarrow t = -5.$$

Therefore,

$$\overrightarrow{OB} = [10 + 2 \cdot (-5), 2 - (-5), 2 + (-5)] = [0, 7, -3],$$

so that $B(0, 7, -3)$.

(c) Since P_3 is parallel to P_1 , it has equation

$$P_3 : 2x - y + z = d_3$$

for some d_3 to be determined. Just like in (a), $(d_3/2, 0, 0)$ lies on P_3 . Denote the distance between P_1 and P_3 by δ . Since P_1 lies between P_2 and P_3 , by (b) (i) the distance between P_2 and P_3 is $30/\sqrt{6} + \delta$. By the question, $30/\sqrt{6} + \delta = 3\delta$, so that $\delta = 15/\sqrt{6}$. Therefore, by (a) (ii),

$$\frac{|20 - d_3|}{\sqrt{6}} = \frac{15}{\sqrt{6}} \Rightarrow d_3 \in \{5, 35\}.$$

Likewise, the distance between P_2 and P_3 is $3\delta = 45/\sqrt{6}$. Therefore, by (a) (ii),

$$\frac{|-10 - d_3|}{\sqrt{6}} = \frac{45}{\sqrt{6}} \Rightarrow d_3 \in \{-55, 35\}.$$

Therefore, $d_3 \in \{5, 35\} \cap \{-55, 35\} = \{35\}$, i.e. $d_3 = 35$. Hence,

$$P_3 : 2x - y + z = 35.$$

(d) (i) Let Q denote the foot of perpendicular from A to the line BC , so that $Q = A$ if A lies on BC . By Pythagoras' theorem (trivially if $Q \in \{A, B\}$),

$$|BA|^2 = |BQ|^2 + |QA|^2 \geq |BQ|^2.$$

Taking square roots, $|QB| \leq |AB|$. Similarly, $|QC| \leq |AC|$. Since B, Q, C are collinear,

$$|BC| \leq |QB| + |QC| \leq |AB| + |AC|.$$

Remark 2. This is simply the triangle inequality.

(d) (ii) By (d) (i), equality holds if and only if $|QA| = 0$ and Q lies between B and C , i.e. A lies on the line segment BC . In that case, C lies on the line AB , which is the normal from (b) (ii). Since C also lies on P_3 , it is the intersection of the normal with P_3 . The normal has equation

$$[x, y, z] = [10, 2, 2] + t[2, -1, 1] = [10 + 2t, 2 - t, 2 + t], \quad t \in \mathbb{R}$$

while $P_3 : 2x - y + z = 35$. Intersecting the two objects,

$$2(10 + 2t) - (2 - t) + (2 + t) = 35 \Rightarrow t = 5/2.$$

Therefore,

$$\overrightarrow{OC} = [10, 2, 2] + \frac{5}{2}[2, -1, 1] = [15, -1/2, 9/2].$$

Question 9. Parametric Equations

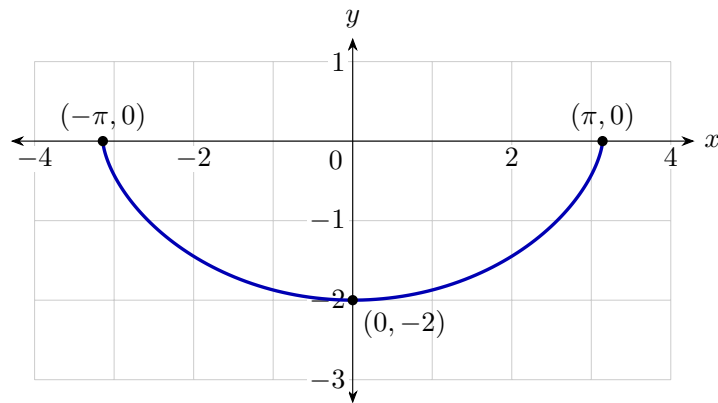
Solution. (a) Using the double angle identity,

$$\sqrt{1 + \cos 2x} = \sqrt{2 \cos^2 x} = \sqrt{2} |\cos x| = \sqrt{2} \cos x,$$

for $-\pi/2 \leq x \leq \pi/2$. Therefore,

$$\int \sqrt{1 + \cos 2x} \, dx = \sqrt{2} \int \cos x \, dx = \sqrt{2} \sin x + C.$$

(b) See the figure below.



(c) (i) Taking derivatives,

$$\begin{aligned} x'(t) &= a((2t)' + (\sin 2t)') \\ &= a(2 + \cos 2t \cdot 2) \\ &= 2a + 2a \cos 2t. \end{aligned}$$

Using the double angle identity again,

$$\begin{aligned} y'(t) &= (-2a \cos^2 t)' \\ &= -2a \cdot ((\cos t)^2)' \\ &= -2a \cdot 2 \cos t \cdot (-\sin t) \\ &= 2a \cdot 2 \sin t \cos t \\ &= 2a \sin 2t. \end{aligned}$$

(c) (ii) Using (c) (i), $\mathbf{v} = 2a[1 + \cos 2t, \sin 2t]$, so that

$$\begin{aligned} \mathbf{v} \cdot \mathbf{v} &= (2a)^2 \cdot ((1 + \cos 2t)^2 + \sin^2 2t) \\ &= 4a^2 \cdot (1 + 2 \cos 2t + \cos^2 2t + \sin^2 2t) \\ &= 4a^2 \cdot (2 + 2 \cos 2t) \\ &= 8a^2 \cdot (1 + \cos 2t). \end{aligned}$$

Taking square roots,

$$\sqrt{\mathbf{v} \cdot \mathbf{v}} = 2a\sqrt{2} \cdot \sqrt{1 + \cos 2t}.$$

By linearity, the required arc length is given by

$$I = \int_{-\pi/2}^{\pi/2} 2a\sqrt{2} \cdot \sqrt{1 + \cos 2t} \, dt = 2a\sqrt{2} \int_{-\pi/2}^{\pi/2} \sqrt{1 + \cos 2t} \, dt.$$

(d) (i) By (a) and (c) (ii),

$$\begin{aligned} I &= 2a\sqrt{2} \cdot \left[\sqrt{2} \sin t \right]_{-\pi/2}^{\pi/2} \\ &= 2a\sqrt{2} \cdot \sqrt{2} \cdot (1 - (-1)) \\ &= 8a. \end{aligned}$$

Since $a = 1$ and the string length is 4 units, we have $8 \cdot 1 = 4k \Rightarrow k = 2$.

(d) (ii) The arc length is given by $I = 8$.

Question 10. Differential Equations

Solution. (a) By the chain rule,

$$\frac{dA}{dX} \cdot \frac{dX}{dt} = \frac{dA}{dt} = -\frac{1}{4} \frac{dX}{dt}.$$

Therefore,

$$\frac{dA}{dX} = -\frac{1}{4} \Rightarrow A = \int -\frac{1}{4} dX = -\frac{1}{4}X + C.$$

Since $A(0) = 2$, we have $2 = 0 + C$ so that $C = 2$, hence

$$A = -\frac{1}{4}X + 2.$$

(b) Substituting the expressions for A and B ,

$$\begin{aligned} \frac{dX}{dt} &= \frac{1}{3} \left(-\frac{1}{4}X + 2 \right) \left(-\frac{3}{4}X + 3 \right) \\ &= \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{3}{4} \cdot (-X + 8) \cdot (-X + 4) \\ &= \frac{1}{16} \cdot (-1) \cdot (X - 8) \cdot (-1) \cdot (X - 4) \\ &= \frac{1}{16} (X - 8)(X - 4). \end{aligned}$$

(c) Using partial fraction decomposition, there exist constants α, β such that

$$\frac{1}{(X - 8)(X - 4)} = \frac{\alpha}{X - 8} + \frac{\beta}{X - 4}.$$

Clearing denominators,

$$1 = \alpha(X - 4) + \beta(X - 8).$$

Setting $X = 8$ yields $\alpha = 1/4$, and setting $X = 4$ yields $\beta = -1/4$, so that

$$\frac{1}{(X - 8)(X - 4)} = \frac{1/4}{X - 8} + \frac{-1/4}{X - 4} = \frac{1}{4} \left(\frac{1}{X - 8} - \frac{1}{X - 4} \right).$$

(d) (i) By the method of separable variables,

$$\int \frac{1}{(X - 8)(X - 4)} dX = \int \frac{1}{16} dt = \frac{1}{16}t + C_1.$$

To evaluate the left-hand side, we use (c) and linearity:

$$\begin{aligned}
\int \frac{1}{(X-8)(X-4)} dX &= \frac{1}{4} \int \left(\frac{1}{X-8} - \frac{1}{X-4} \right) dX \\
&= \frac{1}{4} (\ln |X-8| - \ln |X-4|) + C_2 \\
&= \frac{1}{4} \ln \left| \frac{X-8}{X-4} \right| + C_2.
\end{aligned}$$

Abbreviating $C := C_1 - C_2$,

$$\begin{aligned}
\frac{1}{4} \ln \left| \frac{X-8}{X-4} \right| &= \frac{1}{16} t + C \\
\left| \frac{X-8}{X-4} \right| &= e^{\frac{1}{4}t + 4C} \\
\frac{X-8}{X-4} &= \pm e^{4C} e^{\frac{1}{4}t} =: K e^{\frac{1}{4}t} \\
X(1 - K e^{\frac{1}{4}t}) &= 4(2 - K e^{\frac{1}{4}t}) \\
X &= \frac{4(2 - K e^{\frac{1}{4}t})}{1 - K e^{\frac{1}{4}t}}.
\end{aligned}$$

Using $X(1) = 1$,

$$\frac{1-8}{1-4} = K e^{\frac{1}{4}} \quad \Rightarrow \quad K = \frac{7}{3} e^{-\frac{1}{4}}.$$

Therefore,

$$\begin{aligned}
X &= \frac{4 \left(2 - \frac{7}{3} e^{-\frac{1}{4}} \cdot e^{\frac{1}{4}t} \right)}{1 - \frac{7}{3} e^{-\frac{1}{4}} \cdot e^{\frac{1}{4}t}} \\
&= \frac{4 \left(6 - 7 e^{\frac{1}{4}(t-1)} \right)}{3 - 7 \cdot e^{\frac{1}{4}(t-1)}} \\
&= \frac{24 - 28 e^{\frac{1}{4}(t-1)}}{3 - 7 e^{\frac{1}{4}(t-1)}}.
\end{aligned}$$

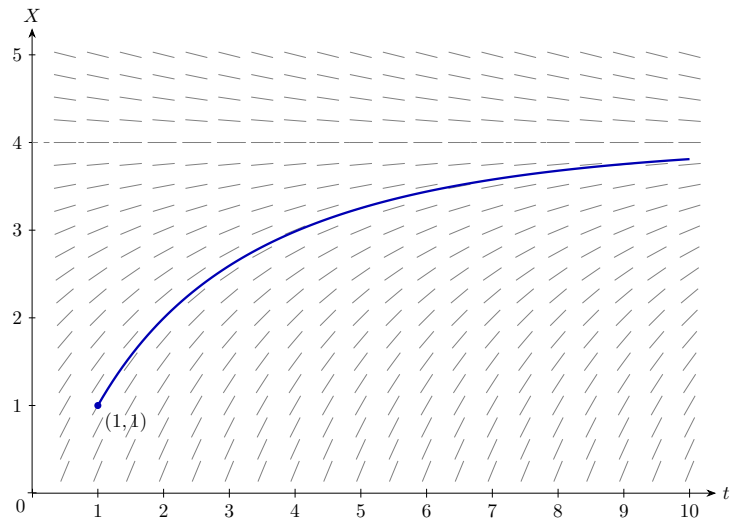
(d) (ii) Writing $24 = 12 + 12$,

$$\begin{aligned}
X &= \frac{12 - 28 e^{\frac{1}{4}(t-1)} + 12}{3 - 7 e^{\frac{1}{4}(t-1)}} \\
&= \frac{4(3 - 7 e^{\frac{1}{4}(t-1)}) + 12}{3 - 7 e^{\frac{1}{4}(t-1)}} \\
&= 4 + \frac{12}{3 - 7 e^{\frac{1}{4}(t-1)}}.
\end{aligned}$$

(d) (iii) We note that as $t \rightarrow \infty$, we have

$$X \rightarrow 4 + 0 = 4.$$

(e) See the figure below.



Remark 3. The diagrams have been drawn and the solutions have been verified with the help of Generative AI.

—Joel Kindiak, 15 Oct 26, 1234H