# 2026 GCE A-Level H2 Mathematics Paper 1
# Examination Solutions

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## Question 1. Recurrence Relations

*Solution.* Setting $n = 1, 2, 3$ in the recurrence relation,

$$\begin{aligned} u_2 = Au_1 + B + C \quad &\Rightarrow \quad 8A + B + C = 22, \\ u_3 = Au_2 + 2B + C \quad &\Rightarrow \quad 22A + 2B + C = 49, \\ u_4 = Au_3 + 3B + C \quad &\Rightarrow \quad 49A + 3B + C = 102. \end{aligned}$$

Solving this system of linear equations using the GC, $A = 2$, $B = -1$ and $C = 7$. Therefore,

$$u_5 = Au_4 + 4B + C = 2(102) + 4(-1) + 7 = 207.$$

## Question 2. Geometric Series

*Solution.* **(a)** The distances run in successive $15$-minute intervals form a geometric sequence with first term $2.5$ and common ratio $0.95$. The interval between $1$ hour and $1$ hour $15$ minutes after the start is the $5$th interval, so she runs a distance of

$$2.5 \times 0.95^4 = 2.036265625 \approx 2.04 \text{ km}.$$

**(b)** The half-way point is at $26$ km. Let $S_n$ km be the distance she has run after $n$ intervals. She reaches the half-way point in the $n$th interval, where $n$ is the least integer such that $S_n \geq 26$. Since $\ln 0.95$ is negative, dividing by it flips the inequality:

$$\begin{aligned} S_n \geq 26 \quad &\iff \quad \frac{2.5(1 - 0.95^n)}{1 - 0.95} \geq 26 \\ &\iff \quad 50(1 - 0.95^n) \geq 26 \\ &\iff \quad 0.95^n \leq 0.48 \\ &\iff \quad n \geq \frac{\ln 0.48}{\ln 0.95} \approx 14.3. \end{aligned}$$

Hence $n = 15$, since $S_{14} \approx 25.6 < 26 \leq 26.8 \approx S_{15}$. The $15$th interval runs from $14 \times 15 = 210$ to $225$ minutes after the start, so she reaches the half-way point between $3$ hours $30$ minutes and $3$ hours $45$ minutes after the start.

**(c)** Since $0 < 0.95 < 1$, the total distance $S_n$ increases towards the sum to infinity

$$S_\infty = \frac{2.5}{1 - 0.95} = 50 < 52.$$

Thus $S_n < 50 < 52$ for every $n$, so according to the model, she will never complete the ultramarathon, however long she runs. In reality, athletes do complete such races, since a runner does not keep slowing down by a constant factor indefinitely. Therefore, the model is not suitable.

## Question 3. Vectors

*Solution.* **(a)** Since $A$, $B$ and $C$ are distinct, the vectors $\overrightarrow{BA} = \mathbf{a} - \mathbf{b}$ and $\overrightarrow{CA} = \mathbf{a} - \mathbf{c}$ are non-zero. Their cross product is zero, so they are parallel. Since they share the point $A$, the points $A$, $B$ and $C$ are collinear.

**(b)** Since $\mathbf{c} - \mathbf{d} = \overrightarrow{DC}$ and $\mathbf{b}$ is a unit vector, $|\mathbf{b} \cdot (\mathbf{c} - \mathbf{d})|$ is the length of the projection of $\overrightarrow{DC}$ onto the line $OB$.

**(c)** Since $\mathbf{b}$ is a unit vector and $OA$ is perpendicular to $OB$,

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} = 2p - 3q + 4q = 2p + q = 0 \quad &\Rightarrow \quad q = -2p, \\ |\mathbf{b}|^2 = p^2 + q^2 + q^2 = p^2 + 8p^2 = 1 \quad &\Rightarrow \quad p = \pm\frac{1}{3}. \end{aligned}$$

Therefore, $p = 1/3$ and $q = -2/3$, or $p = -1/3$ and $q = 2/3$.

## Question 4. Differential Equations

*Solution.* **(a)** Since $y = vx$, the product rule yields $\dfrac{\mathrm{d}y}{\mathrm{d}x} = v + x\dfrac{\mathrm{d}v}{\mathrm{d}x}$. Substituting into the differential equation, and then dividing both sides by $x^3$, where $x \neq 0$,

$$\begin{aligned} x^2\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) &= v^2x^2 + vx^2 \\ x^3\frac{\mathrm{d}v}{\mathrm{d}x} &= v^2x^2 \\ x\frac{\mathrm{d}v}{\mathrm{d}x} &= v^2. \end{aligned}$$

**(b)** Separating the variables, and then substituting $v = y/x$,

$$\begin{aligned} \int \frac{1}{v^2}\,\mathrm{d}v &= \int \frac{1}{x}\,\mathrm{d}x \\ -\frac{1}{v} &= \ln|x| + c \\ -\frac{x}{y} &= \ln|x| + c \\ y &= \frac{x}{-c - \ln|x|} = \frac{x}{A - \ln|x|}, \end{aligned}$$

where $c$ is an arbitrary constant and $A = -c$.

**Remark 1.** The function $y = 0$, which corresponds to $v = 0$, also satisfies the differential equation. It is lost when we divide by $v^2$ to separate the variables.

## Question 5. Complex Numbers

*Solution.* The second equation yields $w = z - 4 + 4i$. Substituting into the first equation,

$$\begin{aligned} 2z + 7i &= -i(z - 4 + 4i) = -iz + 4 + 4i \\ (2+i)z &= 4 - 3i \\ z &= \frac{4-3i}{2+i} \cdot \frac{2-i}{2-i} \\ &= \frac{8 - 4i - 6i + 3i^2}{5} \\ &= \frac{5 - 10i}{5} = 1 - 2i. \end{aligned}$$

Therefore, $z = 1 - 2i$ and $w = z - 4 + 4i = -3 + 2i$.

## Question 6. Inequalities

*Solution.* **(a)** Since the sign of $x + 1$ is unknown, we do not multiply across. Instead,

$$\begin{aligned} \frac{2}{x+1} - (3 - 2x) &< 0 \\ \frac{2 - (3-2x)(x+1)}{x+1} &< 0 \\ \frac{2x^2 - x - 1}{x+1} &< 0 \\ \frac{(2x+1)(x-1)}{x+1} &< 0. \end{aligned}$$

The critical values are $-1$, $-1/2$ and $1$. Checking the sign of the left-hand side on each interval, the inequality holds for

$$x < -1 \quad \text{or} \quad -\frac{1}{2} < x < 1.$$

**(b)** Replacing $x$ with $\sin\theta$, we need $\sin\theta < -1$, which has no solutions, or $-1/2 < \sin\theta < 1$. For $0 < \theta < 2\pi$, we have $\sin\theta > -1/2$ exactly when $0 < \theta < 7\pi/6$ or $11\pi/6 < \theta < 2\pi$, and $\sin\theta < 1$ exactly when $\theta \neq \pi/2$. Therefore,

$$0 < \theta < \frac{\pi}{2}, \quad \frac{\pi}{2} < \theta < \frac{7\pi}{6} \quad \text{or} \quad \frac{11\pi}{6} < \theta < 2\pi.$$

## Question 7. Integration by Parts

*Solution.* **(a)** By the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{\cos 2x}\right) = -2\sin 2x\, e^{\cos 2x}.$$

**(b)** By the double angle formula $\sin 4x = 2\sin 2x \cos 2x$, the integrand is $-\cos 2x$ multiplied by the derivative

in **(a)**. Integrating by parts,

$$\begin{aligned}\int_{\pi/4}^{\pi/2} e^{\cos 2x}\sin 4x\,\mathrm{d}x &= \int_{\pi/4}^{\pi/2}(-\cos 2x)\cdot\left(-2\sin 2x\,e^{\cos 2x}\right)\mathrm{d}x\\ &= \left[-\cos 2x\,e^{\cos 2x}\right]_{\pi/4}^{\pi/2} - \int_{\pi/4}^{\pi/2} 2\sin 2x\,e^{\cos 2x}\,\mathrm{d}x\\ &= \left[-\cos 2x\,e^{\cos 2x}\right]_{\pi/4}^{\pi/2} + \left[e^{\cos 2x}\right]_{\pi/4}^{\pi/2}\\ &= \left[(1-\cos 2x)\,e^{\cos 2x}\right]_{\pi/4}^{\pi/2}\\ &= 2e^{-1} - 1 = \frac{2}{e} - 1.\end{aligned}$$

**Remark 2.** The answer $2/e - 1 \approx -0.264$ is negative, as expected, since $\sin 4x < 0$ for $\pi/4 < x < \pi/2$. It agrees with the value given by the GC.

## Question 8. Rational Functions

*Solution.* **(a)** Rearranging $y(2x-1) = 2x^2$ yields the quadratic equation $2x^2 - 2yx + y = 0$ in $x$. For real values of $x$, its discriminant must be non-negative:

$$\begin{aligned}(-2y)^2 - 4(2)(y) \geq 0 \quad &\iff \quad 4y(y-2) \geq 0\\ &\iff \quad y \leq 0 \quad \text{or} \quad y \geq 2.\end{aligned}$$

Furthermore, $x = 1/2$ never satisfies the quadratic equation, since it would give $1/2 = 0$. Therefore, $y$ cannot take values in the set $\{y \in \mathbb{R} : 0 < y < 2\}$.

**(b) (i)** By long division, $4x^2 + 1 = (2x-1)(2x+1) + 2$, so that

$$y = \frac{4x^2+1}{2x-1} = 2x + 1 + \frac{2}{2x-1}.$$

Thus $a = 2$, $b = 1$ and $c = 2$. As $x \to \pm\infty$, $2/(2x-1) \to 0$, and as $x \to 1/2$, $|y| \to \infty$. Therefore, the asymptotes of $D$ are $x = 1/2$ and $y = 2x + 1$.

**(ii)** Observe that

$$2\left(x + \frac{1}{2x-1}\right) + 1 = 2x + 1 + \frac{2}{2x-1}.$$

Therefore, the graph of $y = x + 1/(2x-1)$ is transformed onto $D$ by a stretch parallel to the $y$-axis with scale factor 2, followed by a translation of 1 unit in the positive $y$-direction.

## Question 9. Maclaurin Series

*Solution.* **(a)** Since $a > 0$ and $1 + \sin x > 0$ on the domain, $f(x) = \ln a + \ln(1+\sin x)$. By the chain rule, and then the quotient rule,

$$\begin{aligned}f'(x) &= \frac{\cos x}{1+\sin x},\\ f''(x) &= \frac{-\sin x(1+\sin x) - \cos^2 x}{(1+\sin x)^2}\\ &= -\frac{\sin x + \sin^2 x + \cos^2 x}{(1+\sin x)^2}\\ &= -\frac{1}{1+\sin x},\end{aligned}$$

so that $k = -1$.

**(b)** Writing $f''(x) = -(1+\sin x)^{-1}$, the chain rule yields $f'''(x) = (1+\sin x)^{-2}\cos x$. Setting $x = 0$, we have $f(0) = \ln a$, $f'(0) = 1$, $f''(0) = -1$ and $f'''(0) = 1$. Therefore,

$$f(x) = \ln a + x - \frac{x^2}{2} + \frac{x^3}{6} + \cdots.$$

**(c) (i)** Setting $a = 2$ and using the series in **(b)**,

$$\begin{aligned}\int_0^{1.2} f(x)\,\mathrm{d}x &\approx \int_0^{1.2} \left(\ln 2 + x - \frac{x^2}{2} + \frac{x^3}{6}\right)\mathrm{d}x \\ &= \left[x\ln 2 + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24}\right]_0^{1.2} \\ &\approx 1.35.\end{aligned}$$

**(ii)** Using the GC,

$$\int_0^{1.2} \ln(2+2\sin x)\,\mathrm{d}x \approx 1.32.$$

**(iii)** The Maclaurin series is truncated after the term in $x^3$, so it approximates $f(x)$ well only for $x$ close to $0$. On most of the interval $[0, 1.2]$, $x$ is not small, so the omitted higher-order terms are significant.

**Remark 3.** The next term of the series is $-x^4/12$. Including it changes the estimate in **(c) (i)** by $-1.2^5/60 \approx -0.041$, to about $1.31$, which is much closer to the actual value.

## Question 10. Functions

*Solution.* **(a)** For $x \le 3$, $|3-x| = 3-x$, so that $f(x) = e^{3-x}$ is strictly decreasing, and hence one-one. Thus $f^{-1}$ exists when $m \le 3$. If instead $m > 3$, then for $0 < h \le m-3$, both $3-h$ and $3+h$ lie in the domain of $f$, with $f(3-h) = e^h = f(3+h)$. Then $f$ is not one-one, and $f^{-1}$ does not exist. Therefore, the greatest value of $m$ is 3.

**(b)** Let $y = e^{3-x}$, where $x \le 3$. Then $3 - x \ge 0$, so that $y \ge 1$, and

$$3 - x = \ln y \quad \Rightarrow \quad x = 3 - \ln y.$$

Therefore, $f^{-1}(x) = 3 - \ln x$, with domain $\mathrm{D}_{f^{-1}} = \mathrm{R}_f = [1, \infty)$.

**(c)** Since $f(f^{-1}(t)) = t$ for every $t \in \mathrm{D}_{f^{-1}}$, the expression $f(f^{-1}(|x|))$ is defined exactly when $|x| \ge 1$, in which case

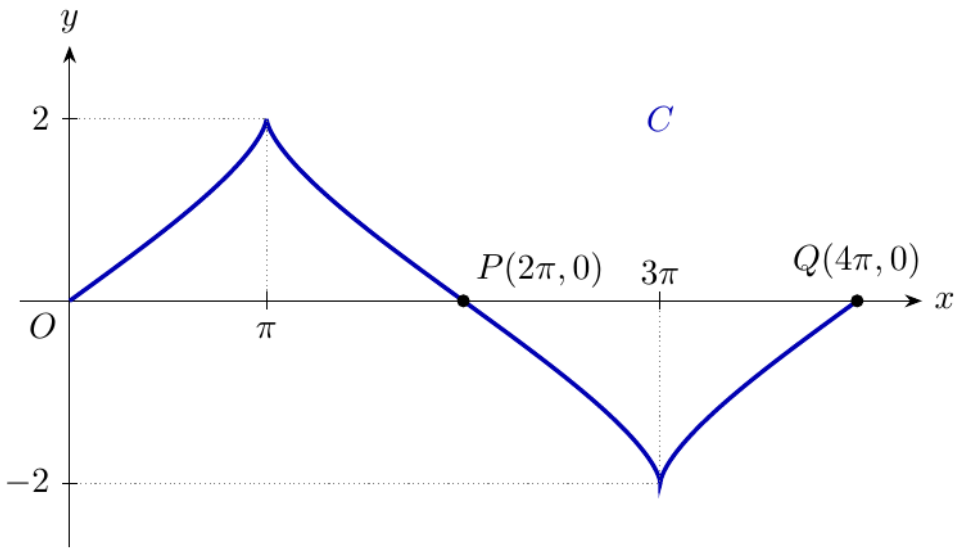
$$f(f^{-1}(|x|)) = |x| \ge 1.$$

Therefore, the solution is $x \le -1$ or $x \ge 1$.

**(d)** The composite function $f^{-1}g$ exists if and only if $\mathrm{R}_g \subseteq \mathrm{D}_{f^{-1}} = [1, \infty)$. Since $\sqrt{x-4}$ takes every value in $[0, \infty)$ for $x \ge 4$, we have $\mathrm{R}_g = [n, \infty)$. Then $[n, \infty) \subseteq [1, \infty)$ if and only if $n \ge 1$, so the smallest value of $n$ is 1.

## Question 11. Parametric Equations

*Solution.* **(a)** We have $y = 0$ exactly when $\sin\theta = 0$, that is, $\theta = 0$, $\pi$ or $2\pi$, giving the points $(0, 0)$, $(2\pi, 0)$ and $(4\pi, 0)$. The value of $y = 2\sin\theta$ changes sign at $\theta = \pi$, so $C$ crosses the $x$-axis at $P(2\pi, 0)$. The curve ends at $\theta = 2\pi$, where it meets the $x$-axis at $Q(4\pi, 0)$.




**(b) (i)** Using the double angle formula $\cos 2\theta = 2\cos^2\theta - 1$, and for $\cos\theta \neq 0$,

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = 2 + 2\cos 2\theta = 4\cos^2\theta,$$
$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = 2\cos\theta,$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}\theta}{\mathrm{d}x/\mathrm{d}\theta} = \frac{2\cos\theta}{4\cos^2\theta} = \frac{1}{2}\sec\theta.$$

**(ii)** Since $|\sec\theta| = 1/|\cos\theta| \geq 1$, the gradient $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{2}\sec\theta$ is never zero. Therefore, the graph of $C$ has no stationary points.

**(iii)** At $\theta = \pi/3$, we have $x = 2\pi/3 + \sqrt{3}/2$, $y = \sqrt{3}$, and the gradient is $(1/2)\sec(\pi/3) = 1$. The tangent therefore has equation

$$y - \sqrt{3} = x - \left(\frac{2\pi}{3} + \frac{\sqrt{3}}{2}\right)$$
$$y = x - \frac{2\pi}{3} + \frac{\sqrt{3}}{2}.$$

Setting $y = 0$ and $x = 0$ in turn, the tangent meets the axes at $R(2\pi/3 - \sqrt{3}/2, 0)$ and $S(0, \sqrt{3}/2 - 2\pi/3)$. Hence $OR = OS = 2\pi/3 - \sqrt{3}/2$, and by Pythagoras' theorem,

$$RS = \sqrt{2}\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) = \frac{\sqrt{2}(4\pi - 3\sqrt{3})}{6} \approx 1.74.$$

## Question 12. Planes

*Solution.* **(a)** The plane contains the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$, so their cross product is a normal vector to the plane:

$$\begin{aligned}
\overrightarrow{AB} &= \begin{bmatrix} 5 \\ 8 \\ -40 \end{bmatrix} - \begin{bmatrix} 10 \\ 15 \\ -60 \end{bmatrix} = \begin{bmatrix} -5 \\ -7 \\ 20 \end{bmatrix}, \\
\overrightarrow{AC} &= \begin{bmatrix} -5 \\ 10 \\ -50 \end{bmatrix} - \begin{bmatrix} 10 \\ 15 \\ -60 \end{bmatrix} = \begin{bmatrix} -15 \\ -5 \\ 10 \end{bmatrix}, \\
\overrightarrow{AB} \times \overrightarrow{AC} &= \begin{bmatrix} (-7)(10) - (20)(-5) \\ (20)(-15) - (-5)(10) \\ (-5)(-5) - (-7)(-15) \end{bmatrix} = \begin{bmatrix} 30 \\ -250 \\ -80 \end{bmatrix} = 10 \begin{bmatrix} 3 \\ -25 \\ -8 \end{bmatrix}.
\end{aligned}$$

Taking the normal vector $\mathbf{n}$ with components $3$, $-25$ and $-8$, and since $A$ lies on the plane,

$$\mathbf{r} \cdot \mathbf{n} = \begin{bmatrix} 10 \\ 15 \\ -60 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -25 \\ -8 \end{bmatrix} = 30 - 375 + 480 = 135.$$

Therefore, the plane has cartesian equation $3x - 25y - 8z = 135$. As a check, $B$ gives $15 - 200 + 320 = 135$ and $C$ gives $-15 - 250 + 400 = 135$.

**(b)** The ground $z = 0$ has normal vector $\mathbf{k}$. The acute angle $\theta$ between the ground and the mineral layer satisfies

$$\begin{aligned}
\cos\theta &= \frac{|\mathbf{n} \cdot \mathbf{k}|}{|\mathbf{n}||\mathbf{k}|} = \frac{|-8|}{\sqrt{3^2 + 25^2 + 8^2}} = \frac{8}{\sqrt{698}}, \\
\theta &\approx 72.4^\circ.
\end{aligned}$$

**(c)** Points on the vertical line through $P$ have coordinates $(-17, 188, \lambda)$, where $\lambda \in \mathbb{R}$. Substituting into the equation of the plane,

$$\begin{aligned}
3(-17) - 25(188) - 8\lambda &= 135 \\
-4751 - 8\lambda &= 135 \\
\lambda &= -610.75.
\end{aligned}$$

Therefore, the vertical distance from $P$ to the plane is $610.75$ m, or $611$ m to 3 significant figures.

**(d)** Let $F$ be the foot of the perpendicular from $P$ to the plane. Then $\overrightarrow{OF} = \overrightarrow{OP} + \mu\mathbf{n}$ for some $\mu$, and since $F$ lies on the plane,

$$\begin{aligned}
\left(\overrightarrow{OP} + \mu\mathbf{n}\right) \cdot \mathbf{n} &= 135 \\
\overrightarrow{OP} \cdot \mathbf{n} + \mu|\mathbf{n}|^2 &= 135 \\
-4751 + 698\mu &= 135 \\
\mu &= 7.
\end{aligned}$$

Thus $F$ has coordinates $(-17 + 21, 188 - 175, 0 - 56) = (4, 13, -56)$, and the shortest distance from $P$ to the plane is

$$PF = 7|\mathbf{n}| = 7\sqrt{698} = \sqrt{34\,202} \approx 184.9 \text{ m}.$$

**Remark 4.** Alternatively, the shortest distance is the vertical distance in **(c)** multiplied by $\cos\theta$ from **(b)**: $610.75 \times 8/\sqrt{698} \approx 184.9$ m. This is because the vertical line and the perpendicular to the plane meet at the angle $\theta$ between the ground and the mineral layer.

**Remark 5.** The diagrams have been drawn and the solutions have been verified with the help of Generative AI.

—Joel Kindiak, 3 Nov 26, 1430H